Exercises for the lecture History and Geometry of Kepler's Laws.

Problem 1 – stereographic projection and inversion. Imagine a sphere of radius r lying on a plane touching it at a point S. Consider the stereographic projection ρ from the sphere to the plane; recall that if N denotes the point of the sphere furthest from the plane then for a point $X \neq N$ on the sphere $\rho(X)$ is the unique point of intersection of NX with the plane.

Let i be the inversion of the plane with respect to the circle of radius r centered at S and let σ be the π rotation (central symmetry) of the plane centered at S.

Prove that the images of big circles not passing through N under ρ are exactly the circles invariant under $i\sigma$.

In the remaining exercises we sketch other ways to derive Kepler 1st Law from Newton's Laws.

Let $\mathbf{r}(t)$ denote the vector pointing from the Sun to the planet at time t, we'll set $r = |\mathbf{r}|$. Newton's Laws say that (neglecting gravitational forces between the planets etc.)

$$\mathbf{r}'' = -k\frac{\mathbf{r}}{r^3}, \tag{1}$$

where k depends on the mass of the Sun and the gravitational constant (but does not depend on the mass of the planet).

Problem 2 – conservation of energy. Show that $E = \frac{|\mathbf{r}'|^2}{2} - \frac{k}{r}$ is a conserved quantity for (1).

Problem 3 Show that for a periodic planetary motion $E < 0$.

[Remark: As was mentioned in the lecture, the case $E > 0$ is that of a comet traversing an infinite trajectory which is a hyperbola (as long as the approximation neglecting everything but the Sun gravitation is valid.]

Problem 4 – "complex" proof.[1] Show that the map $\mathbb{C} \to \mathbb{C}$, $z \mapsto z^2$ sends an ellipse with center at 0 to an ellipse with a focus at 0.

[Hint: consider the image of a circle centered at zero under the maps $z \mapsto z+z^{-1}$, $z \mapsto z^2 + z^{-2} + 2$].

Show that if $z(t)$ is a solution for $z''(t) = -z$ then $\mathbf{r}(t) = z^2(t)$ is a solution of (1) (with $k = 1$). Deduce the 1st Kepler's Law.

Problem 4 – proof by integration. (best done with some 18.03 or 18.034 experience)

The idea is to use the two conservation laws obtained above to reduce the problem to a 1st order ODE in one variable which is then integrated explicitly. We work in polar coordinates (r, θ).

Use angular momentum conservation to show that $\theta' = \frac{M}{r^2}$ where M is a constant.

Show that $r'' - r(\theta')^2 = -\frac{k}{r^2}$, and then that

$r'' = \frac{k}{r^2} + \frac{M^2}{r^3}$.

Prove that $(r')^2 + V(r)$ where $V(r) = -\frac{k}{r} + \frac{M^2}{r^2}$ is a conserved quantity.

Setting $E_1 = \frac{(r')^2}{2} + V(r)$ obtain the equation $\frac{d\theta}{dr} = \frac{M/r^2}{\sqrt{2(E_1 - V(r))}}$.

[1] see V.I. Arnold "Huygens and Barrow, Newton and Hooke", Appendix 1

Solve it to get $\theta = \arccos\left(\frac{M/r - k/M}{\sqrt{2E_1 + k^2/M^2}}\right) + C$. Deduce that orbits are ellipses.

Problem 5 – another geometric proof $(*)$**.**

We will use the vector product $\mathbf{v} \times \mathbf{w}$ of vectors in $\mathbb{R}^3$.

Set $\mathbf{L} = \mathbf{r} \times \mathbf{r}'$, $\mathbf{K} = \mathbf{r}' \times \mathbf{L} - k\frac{\mathbf{r}}{r}$.

Show that $\mathbf{K}$ is a constant vector for any solution $\mathbf{r}$ of (1). $\mathbf{K}$ is called the Runge-Lenz vector, although it is said to be first discovered by Laplace in 1789.

Let S be the "fall circle" – the locus of points $\mathbf{x} \in \mathbb{R}^2$ such that $\frac{-k}{|\mathbf{x}|} = E$.

For every t consider the ray of $\mathbf{r}(t)$ and let $\mathbf{x}(t)$ be the intersection of this ray with the fall circe. Show that the end-point of $\frac{\mathbf{K}}{E}$ is symmetric to $\mathbf{x}(t)$ with respect to the tangent to the orbit at point t. Deduce that the orbit is an ellipse with foci at the Sun and the end-point of $\frac{\mathbf{K}}{E}$.

Remark: $\mathbf{K}$ can be used to uncover the hidden symmetries of Kepler's problem algebraically via the action of Lie algebra $so(4)$ (about which you can learn in 18.715 and other classes mentioning Lie theory).