HEAT EQUATIONS AND GEOMETRY

Reading material: https://www.ams.org/journals/notices/201610/rnoti-p1148.pdf

1. Background

A function $u(x,t)$ with $x \in \mathbf{R}^n$ satisfies the heat equation if

$$u_t = \Delta\, u = \sum_{i=1}^{n} u_{x_i\, x_i} = \operatorname{div} \nabla u \,.$$

Recall that the divergence theorem gives that if V is a vector field on a bounded domain Ω with boundary $\partial\Omega$, then

$$\int_{\Omega} \operatorname{div} V = \int_{\partial\Omega} V \cdot \nu \,,$$

where ν is the outward pointing unit normal to $\partial\Omega$. The boundary integral is called the flux in 18.02.

2. Exercises

Question 1. Show that $\Delta\, |x|^2 = 2n$.

Question 2. Using the first question, find a quadratic polynomial in x and t that satisfies the heat equation.

Question 3. Suppose that u satisfies the heat equation on a ball B and u_t is zero on the boundary ∂B (i.e., u is constant on the boundary). Prove that $\int_B |\nabla u|^2$ is non-increasing in time.

Question 4. From the reading, what is the level set flow equation?