Triangles & equations

Schur 1910's

$$X^n + Y^n = Z^n \pmod p$$

If you can show that for infinitely many primes p has no non-triv solns $\rightsquigarrow$ ☺

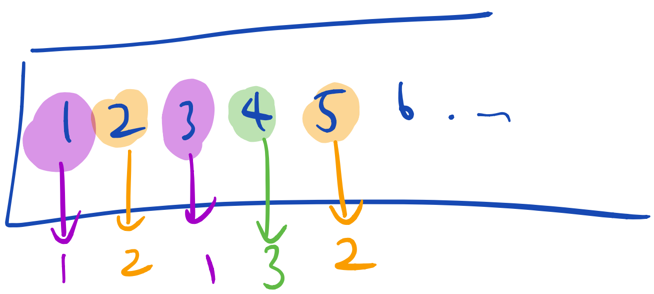
Thm 1 If $\mathbb{N}$ colored with finitely many colors, then there exist a monochromatic soln to $x+y=z$ — infinitary

$\Updownarrow$

Thm 2 For every $r \in \mathbb{N}$, $\exists$ $N = N(r)$ s.t. if $[N] = \{1, 2, \dots, N\}$ colored using r colors, then $\exists$ a monochromatic soln to $x+y=z$ with $x, y, z \in [N]$. — finitary

Pf of $\Leftrightarrow$

Pf of $\Leftarrow$ ✓



Pf of $\Rightarrow$ diagonalization

Suppose Thm 2 false

$\exists r$ s.t. $\forall N$ $\exists$ $\phi_N : [N] \to [r]$ with no monochr. solns to $x+y=z$

can find subsequence $\phi_{N_1}, \phi_{N_2}, \phi_{N_3}, \dots$ s.t. $\phi_{N_i}(x)$ does not depend on i

$\rightsquigarrow$ $\phi : \mathbb{N} \to [r]$

Thm 3 Let $n \in \mathbb{N}$. For all suff. large primes p, $\exists$ $X, Y, Z \in \{1, \dots, p-1\}$ s.t.

$$X^n + Y^n \equiv Z^n \pmod p$$

Pf that Thm 2 ⇒ Thm 3.

$(\mathbb{Z}/p\mathbb{Z})^\times = \{1, 2, \dots, p-1\}$

H = all elts of $(\mathbb{Z}/p\mathbb{Z})^\times$ that can be written as an n-th power of some elt of $(\mathbb{Z}/p\mathbb{Z})^\times$

primitive root $g \in (\mathbb{Z}/p\mathbb{Z})^\times \leftarrow$ cyclic gp $\cong \mathbb{Z}/(p-1)\mathbb{Z}$

$H = \{g^{ni} : i \in \mathbb{Z}\}$

H is a subgroup of $(\mathbb{Z}/p\mathbb{Z})^\times$

Cosets of H partitions $(\mathbb{Z}/p\mathbb{Z})^\times$

ie. $\exists\, \underline{a_1, a_2, \dots, a_k}$ st. every elt of $(\mathbb{Z}/p\mathbb{Z})^\times$ can be written as $a_i h$ for some unique choices of $i \in [k]$ & $h \in H$

$k \le n$

Coloring of $\mathbb{Z}/p\mathbb{Z}$ based on which coset it's in

ie. color $a_i h \in (\mathbb{Z}/p\mathbb{Z})^\times$ by the color i

⇝ We're coloring $\{1, \dots, p-1\}$ using n colors

By Thm 2, if p is large enough, then $\exists$ monochromatic

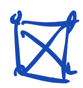
$x + y = z$ in $\mathbb{Z}$

$\cancel{a_i} X^n + \cancel{a_i} Y^n = \cancel{a_i} Z^n \mod p$ for some $X, Y, Z \in (\mathbb{Z}/p\mathbb{Z})^\times$

since $a_i \in (\mathbb{Z}/p)^\times$

Thm 4 $\forall r\ \exists N = N(r)$ s.t.

if edges of K_N ($\leftarrow$ complete graph on N vtx) are colored using r colors, then there is always a monochromatic triangle.

$K_4 =$ 

$\triangle$

Ramsey

Among 6 ppl, $\exists$ 3 mutual friends
or $\exists$ 3 mutual non-friends

Among disorder there is always some structure

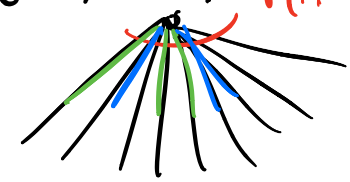
Pf Induction on r.

$N(1) = 3$ works

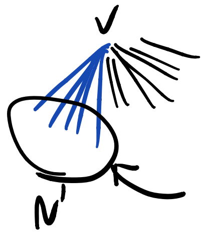
Claim: if thm true for $r-1$ colors & N'
then true for r & $N = r(N'-1)+2$

Suppose we color complete graph on $r(N'-1)+2$ vtx using r colors

Pick arb vtx v. $r(N'-1)+1$ edges



By pigeonhole, some color has
$\geq N'$ edges coming from v with that color



if blue edge here (among blue edge neighbors of v)
then found blue Δ
else N' vtx & $\leq r-1$ colors
so by induction, $\exists$ monochro. triangle here //

$[N] \longleftrightarrow$ graph

$x+y=z \longleftrightarrow$ triangles

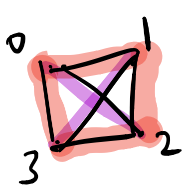
Pf Thm 4 $\Rightarrow$ Thm 2

$\phi : [N] \to [r]$ coloring of $[N] = \{1, 2, \ldots, N\}$

K_{N+1} vtx $\{0, 1, 2, \ldots, N\}$

color edge $i \sim j$ using the color $\phi(|i-j|)$

1 2 3



By Thm 4, if N large enough, then $\exists$ monochr. triangle colored red in ϕ

$$\overbrace{\underbrace{i \;<\; j}_{x=j-i} \;\;\underbrace{<\; k}_{y=k-j}}^{z=k-i}$$

$x+y=z$ ← monochr. soln

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Additive combinatorics

van der Waerden's thm 1927 If $\mathbb{N}$ colored using finitely many colors, then one of the color class contains arbitrarily long arithmetic progressions

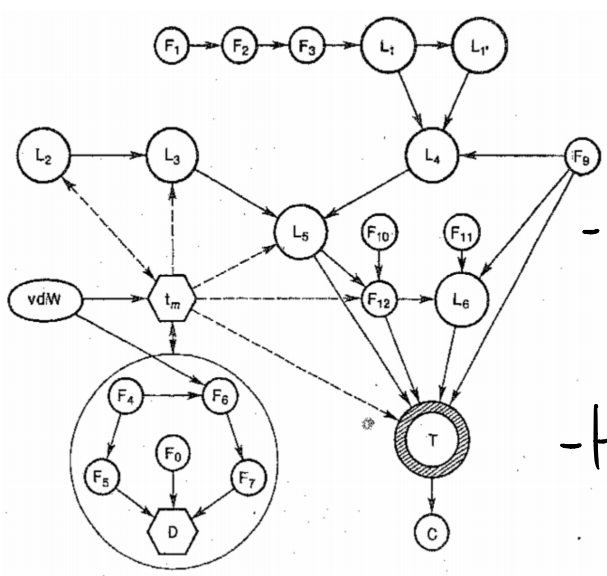
⚠ $\neq$ infinitely long

Roth's theorem (1953) Every subset of integers with positive density contains a 3-term arithmetic progression

$A \subset \mathbb{N}$

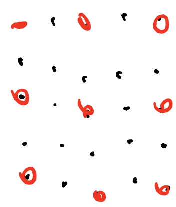
$$\limsup_{N\to\infty} \frac{|A\cap[N]|}{N} > 0$$

Szemerédi's theorem (1975) k-term $\forall k$



- Ergodic theory
- Higher-order Fourier analysis
- Hypergraph regularity

$A \subset \mathbb{Z}^2$



square $k\times k$ subgrid

Dichotomy between structure vs. randomness

Green-Tao thm The primes contain arb long APs.

$$\#\{\text{primes} \le N\} \sim \frac{N}{\log N} \quad (\text{prime \# thm})$$

Q1 What's max # edges in an n-vtx without triangles?

$\left\lfloor \frac{n^2}{4} \right\rfloor$

Q2 What's max # edges in an n-vtx s.t. every edge is contained in exactly one triangle?

$$n^{2-o(1)} < \quad < o(n^2)$$