The Heat Equation

1-D. $u(x,t) = $ Temperature at pt $x \in \mathbb{R}$ at time t

x: space, t: time

$$(*) \quad \boxed{\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}}$$

write this as $u_t = u_{xx}$

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n-D. $u(\underbrace{x_1, \ldots, x_n}_{\text{space}}, t)$

$$u_t = \sum_{i=1}^{n} u_{x_i x_i} \equiv \Delta u$$

Δ: The "Laplacian"

2-D $\quad \Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$

Smoothing:

$u_t = u_{xx}$ on $\boxed{[0, \pi]}$ $\Longrightarrow$

$u = 0$ at endpts

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin(nx)\, e^{-n^2 t}$$

[exponential decay in t]

$$\partial_t \left[e^{-n^2 t} \sin nx \right] = -n^2 [\quad]$$

$$\partial_x^2 \quad = \text{same}$$

$$u(x,0) = \sum a_n \sin nx$$

$$u(x,1000) = \sum a_n \sin nx\, e^{-1000 n^2}$$

small

Energy Decay

$u_t = u_{xx}$ on $[0, \pi]$

$$\underset{E(t)}{\text{Energy}} = \int_0^\pi u_x^2 \, dx$$

$u_t = 0$ at $0, \pi$

Compute $E'(t) =$

$$2\int_0^\pi u_x u_{xt} \, dx$$

justify

$$= -2\int_0^\pi u_{xx} u_t \, dx$$

$u_t = u_{xx} \Rightarrow$

$$= -2\int_0^\pi u_t^2 \, dx \leq 0$$

Maximum Principle:

$u_t = u_{xx}$ on $[0, \pi]$

$u = 0$ at $0, \pi$

Def:

$$M(t) = \max_x u(x, t).$$

$$\boxed{\text{If } M(t) > 0, \text{ then } M(t) \searrow}$$

Sketch:

Heat Eq'n in Geometry

curve shortening flow

MCF

Ricci Flow

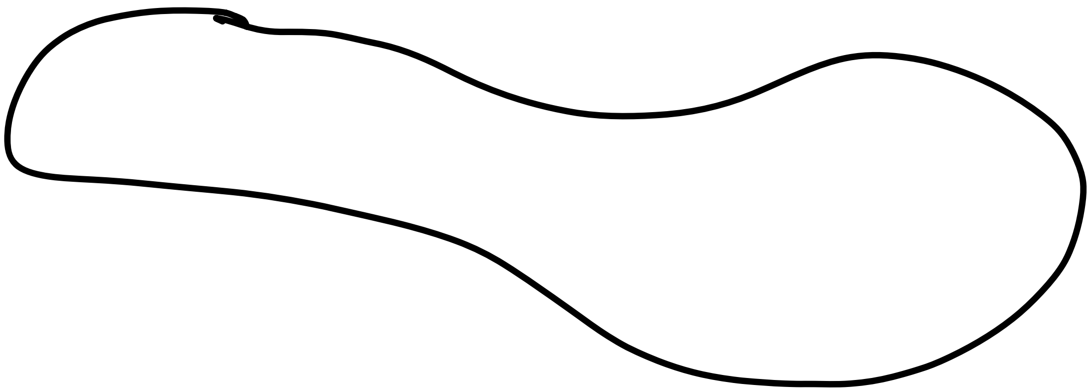
Harmonic Map Heat Flow

Inverse MCF — GR

Mfld, M Metric g

$$g_t = -2\,Ric_g$$

Curve Shortening



$\gamma(s,t) \in \mathbb{R}^2$ normal $\vec{n}$

$$\gamma_t = -k\,\vec{n}$$

curvature

$k \equiv \langle -\gamma_{ss}, \vec{n} \rangle$ when $|\gamma_s| = 1$

(arclength)

Level set

$$\gamma = \{u = c\}$$

$$\kappa = \frac{\mathrm{Hess}_u}{|\nabla u|}(\gamma_s, \gamma_s)$$

$$= \frac{\Delta u}{|\nabla u|} - \frac{\mathrm{Hess}_u}{|\nabla u|}\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right)$$

$$\doteq \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right)$$

$$\frac{\nabla u}{|\nabla u|} = \vec{n}$$ is unit normal to level set.

Arrival time

$u(x)$ = time that γ goes through x.

$$\boxed{-1 = |\nabla u| \, \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right)}$$

Same eq'n in all dim's.

Degenerate Elliptic PDE

Level sets of u describe the evolving curves

Energy of a function:

(Dirichlet) $E(u)$

Const should
$E(\text{const}) = 0$

$\boxed{E(u) \text{ measures how far from constant.}}$

$u_x = 0 \iff$ const.

$$E(u) = \int_0^\pi |u_x|^2$$

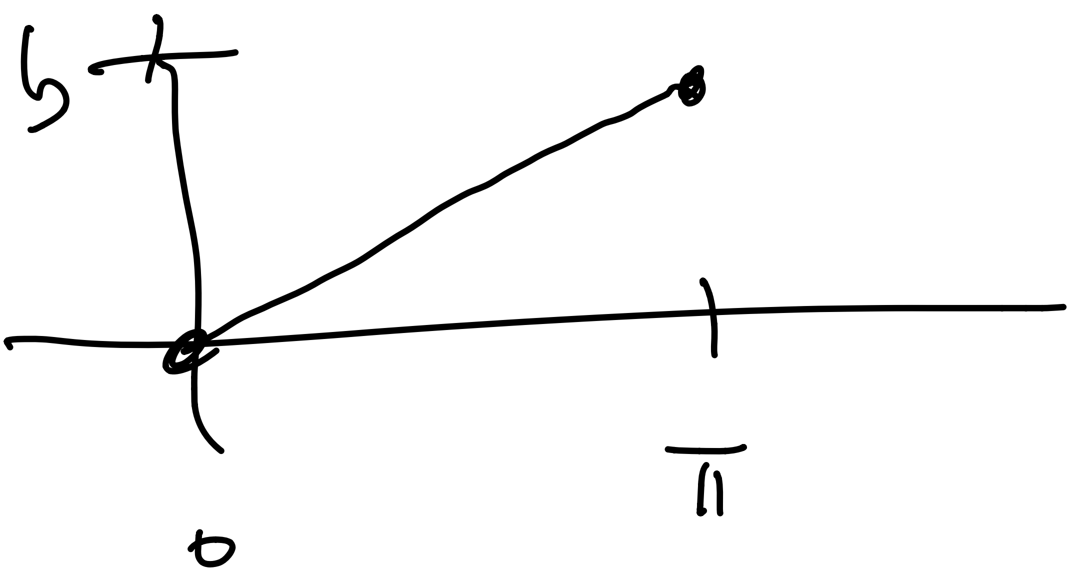
arbitrary, but smart, choice.

Fix bdry values

$u(0) = a$

$u(\pi) = b$

Find u with least $E(u)$



v any other fn,
that 0 at $0, \pi$

$u + sv$ any s
has right bdry values.

If u was best,

$$E(u+sv) \geq E(u)$$

$$\left.\frac{d}{ds}\right|_{s=0} E(u+sv) = 0$$

$$\frac{d}{ds}\int_0^{\pi}(u_x+sv_x)^2$$

$$=\frac{d}{ds}\int_0^{\pi}\left[u_x^2+2s\,u_xv_x+s^2v_x^2\right]$$

no s

$$=\int_0^{\pi}\left[2u_xv_x+2s\,v_x^2\right]$$

At $s=0$

bye-bye

$$\frac{d}{ds}\Big|_{s=0} E(u+sv) = 0$$

$$\parallel \qquad\qquad \parallel$$

$$2\int_0^\pi u_x v_x$$

$v u_x = 0$ at endpts

$$0 = \int_0^\pi (v u_x)_x = \int_0^\pi v_x u_x + v u_{xx}$$

0 $\Rightarrow$ also zero

$\Rightarrow$ u minis, then

$$\boxed{\int_0^{\pi} v\, u_{xx} = 0}$$

for any v

$$\Rightarrow \boxed{u_{xx} = 0}$$

$u_x = \text{Const}$

$\Rightarrow$ u is linear

gradient flows

u_{xx} is the "gradient of E"

$u_t = u_{xx}$ is a gradient flow that decreases E as rapidly as possibly.

E-man's in n-D

$\Delta u = 0$

$(x^2 - y^2)$ on $\mathbb{R}$

harmonic.

$$\boxed{u_t = \Delta u}$$

$\lim_{t \to \infty}$ becomes harmonic

$$\frac{d}{ds}\Big|_{s=0} E(u+sv)$$

$$= -2\int v\, u_{xx}$$

Choose v to make

$E \downarrow$ best possible?

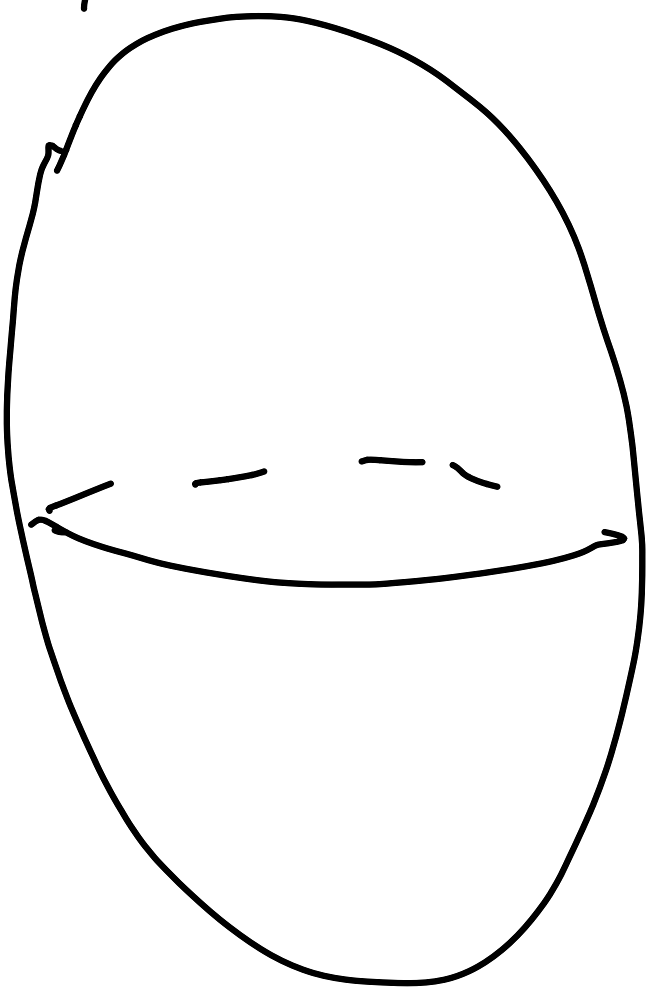
$v = u_{xx}$!!

$$\boxed{\text{Way to change } u \text{ move in } u_{xx} \text{ direction}}$$

$$u_t = u_{xx}$$

Higher Dim's

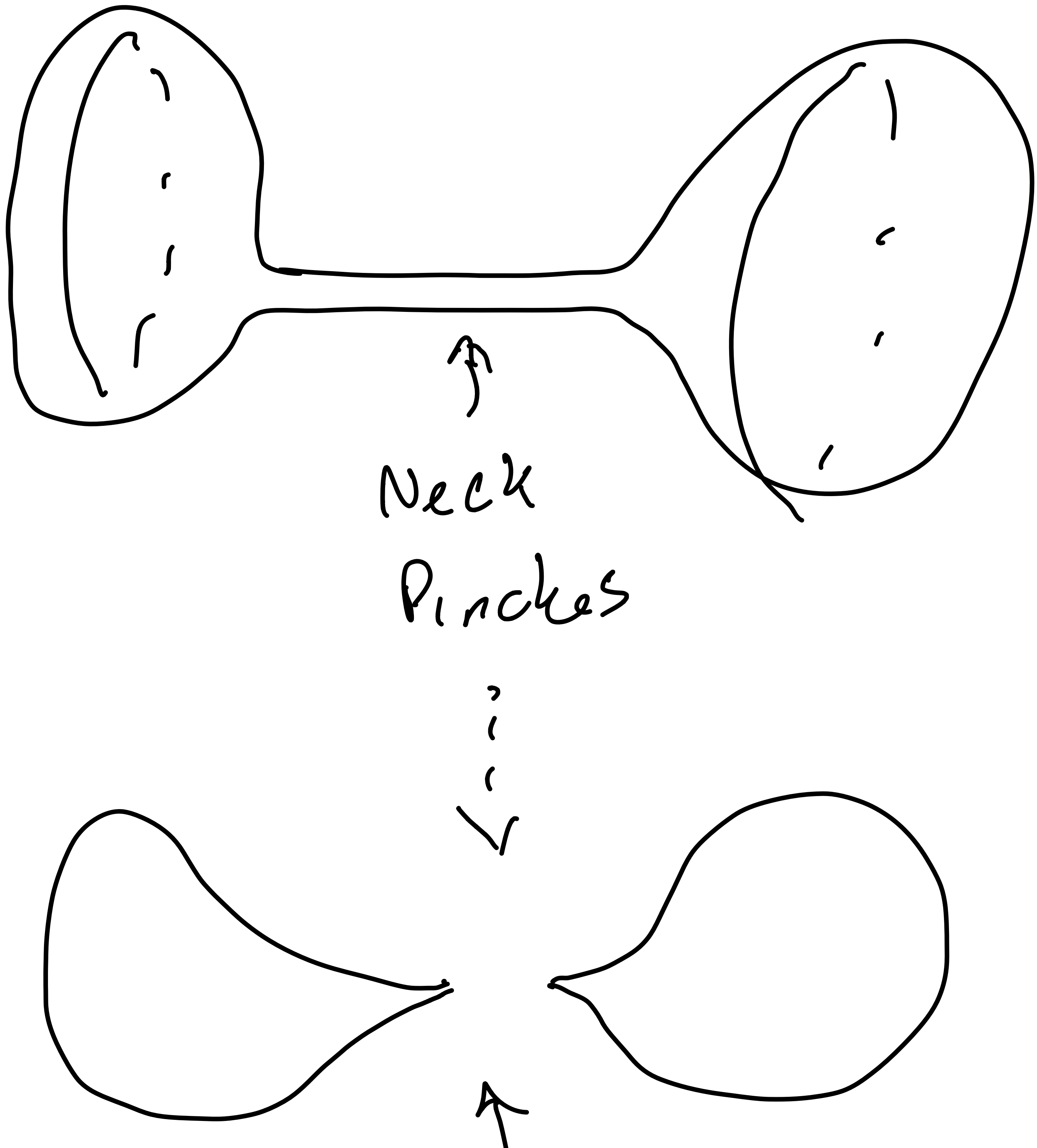
Cvx → Same



→ Disap. at Round pt.

Huisken (dim = 2 ~~& 0~~ but not 1)

Grayson Fails horribly.



Disconnected
at "neck pinch"

Blow up Analysis At
Sing's

Look just B4
neck pinch +
magnify

looks like cylinder