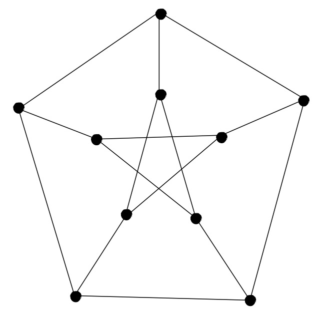
Problem set for 2021 IAP lecture on "How to cut a graph" by Michel Goemans.

1. For the Petersen graph (see below), find a maximum cut and also numerically compute the eigenvalue upper bound described in lecture.



   As a reminder, given the Laplacian L of a graph $G = (V, E)$ (defined as a matrix with rows and columns indexed by vertices, with the degrees along the diagonal and an entry -1 for row i and column j if there is an edge (i, j)), we have that $\frac{|V|}{4} \max_i \lambda_i(L)$ is an upper bound on the size of the maximum cut.

   To compute the eigenvalue bound numerically, you can use MATLAB for example (it is available for the MIT community through IS&T).

   If you have never used MATLAB before, here is a basic tutorial `http://web.mit.edu/18.06/www/MATLAB/matlab.pdf` written for 18.06.

2. For the k-cycle (a cycle on k vertices, $k \geq 3$), give a closed-form formula for the eigenvalue bound as a function of k. (Should be trivial for k even.)

3. In this problem, you will prove the main part of Menger's theorem which says that in a graph $G = (V, E)$ with 2 specified vertices s and t, the maximum number of edge-disjoint (with no edges in common) paths between s and t is equal to $\lambda(s, t)$ (i.e. the minimum number of edges in any cut separating s and t). We saw in lecture that the maximum is always at most the minimum (since every s-t path must cross any s-t cut), so we'll focus on the reverse inequality.

   Suppose we have found the maximum number, say k, of edge-disjoint paths from s to t. Let P_1, P_2, $\cdots$, P_k be these paths. If we can exhibit a cut of size k we are done as this would show $k \geq \lambda(s, t)$.

   Construct a *directed* graph $H = (V, A)$ in the following way. Direct the paths $P_1, P_2, \cdots, P_k$ from t to s. For any edge (u, v) not in any these paths, replace (u, v) by two directed edges, one from u to v (denoted by (u, v)) and the other from v to u (denoted by (v, u)).

   (a) Argue that H has no directed path (i.e. a path which agree with the direction of all its edges) from s to t since, otherwise, you could find (show how) a set of $k+1$ edge-disjoint paths from s to t in G, contradicting the maximality of k.

   (b) In H, define S to be the set of vertices reachable by a directed path from s. Argue that the cut $(S, \bar{S})$ has exactly k edges, concluding the proof of Menger's theorem.