IAP - Math Lecture Series: January 2021

I. Shape of a 2D meniscus

Consider the meniscus resulting from the interaction between a solid and a free surface (Figure 1). Assume that the contact angle, θ, is prescribed. The normal force balance at the interface is expressed by the Young-Laplace equation:

$$\rho g z = \sigma \nabla \cdot \mathbf{n}, \tag{1}$$

which expresses the balance between hydrostatic and curvature pressures. Here ρ is the fluid density, $\vec{g} = -g\ \hat{\mathbf{z}}$ is gravitational acceleration, $z = h(x)$ is the height of the free surface, σ is the surface tension. $\mathbf{n}$ is the unit normal to the surface pointing towards air, and $\nabla \cdot \mathbf{n}$.

a) By defining a functional $f(x, y, z) = z - h(x, y)$ that vanishes on the free surface, show that the unit normal $\mathbf{n} = \frac{\nabla f}{|\nabla f|}$ may be expressed

$$\mathbf{n} \ = \ \frac{\hat{\mathbf{z}} - h_x \hat{\mathbf{x}}}{(1 + h_x^2)^{1/2}} \quad , \tag{2}$$

and that the curvature of the surface may be expressed as

$$\nabla \cdot \mathbf{n} \ = \ \frac{-h_{xx}}{(1 + h_x^2)^{3/2}} \quad . \tag{3}$$

b) Now by assuming that the interfacial slope is relatively small $h_x^2 \ll 1$, one has that $\nabla \cdot \mathbf{n} \ = \ -h_{xx}$. By substituting this into the Young-Laplace equation (1), solve the resulting differential equation for the interface shape $h(x)$. By applying appropriate boundary conditions, i) $h \to 0$ as $x \to \infty$, and ii) $h_x = -\cot\theta$ at $x = 0$, show that the meniscus shape is given by:

$$h(x) \ = \ L_c \cot\theta \exp(-x/L_c) \tag{4}$$

Note that the capillary length, $L_c = \sqrt{\frac{\sigma}{\rho g}}$, prescribes the characteristic length over which the meniscus relaxes to horizontal.

II. Forces on Floating Bodies

Floating bodies must be supported by some combination of buoyancy and curvature forces. Specifically, since the fluid pressure beneath the interface is given by

$$p \ = \ P_0 \ + \ \rho g z \ + \ \sigma \nabla \cdot \mathbf{n} \ = \ P_0 \ + \ p_g \ + \ p_c \ \ ,$$

one may calculate the total hydrostatic force on the body by integrating the hydrostatic pressure p_g and curvature pressure p_c over the surface of the body.

The vertical force balance on the body may thus be expressed:

$$Mg \;=\; \hat{\mathbf{z}} \cdot \int_{C_1} -\, p \;\mathbf{n}\; ds \;=\; F_b \;+\; F_c \quad , \tag{5}$$

where F_b and F_c are respectively the buoyancy and curvature forces, defined below.

a) By using the Divergence Theorem, show that the buoyancy force

$$F_b \;=\; \hat{\mathbf{z}} \cdot \int_{C_1} \rho g \; z \;\mathbf{n}\; ds \tag{6}$$

is simply expressible in terms of the volume (per unit length into the page) V_c of fluid displaced above the object and inside the line of tangency (Figure 2). Thus prove Archimedes Principle holds for a floating body: the buoyancy force is equal to the weight of the fluid displaced by the body: $F_b = \rho V_c g$.

b) Proceed by using the Frenet-Serret equation $(\nabla \cdot \mathbf{n})\;\mathbf{n} \;=\; \frac{d\mathbf{t}}{ds}$, that relates the curvature to the derivative of the unit tangent $\mathbf{t}$ with respect to the arclength s. By integrating the curvature pressure along the line of contact C_1, show that the curvature force may be expressed

$$F_c \;=\; \hat{\mathbf{z}} \cdot \int_{C_1} \sigma(\nabla \cdot \mathbf{n}) \;\mathbf{n}\; ds \;=\; 2\sigma \,\sin\theta \tag{7}$$

c) Show that the relative magnitude of the buoyancy and curvature forces supporting a floating, non-wetting (2D) body is thus proportional to the relative magnitudes of the characteristic body size L and the capillary length L_c:

$$\frac{F_b}{F_c} \;\sim\; \frac{L}{L_c} \quad . \tag{8}$$

For an air-water interface, $g = 980$ cm/s^2, $\rho = 1$g/cc, $\sigma = 70$g/s^2 and the capillary length $L_c \sim 3$mm. Very small objects (with characteristic size $L \ll L_c$), such as paper clips, pins or insects, are thus supported principally by curvature rather than buoyancy forces. They may reside at rest on a free surface provided the curvature force induced by their deflection of the free surface is sufficient to bear their weight.

d) For a body of contact length X and total mass M supported by surface tension, show that static equilibrium on the free surface requires that:

$$\frac{Mg}{\sigma X \sin\theta} \;<\; 1 \tag{9}$$

where θ is the angle of tangency of the floating body. How big must your feet be for you to walk on water?

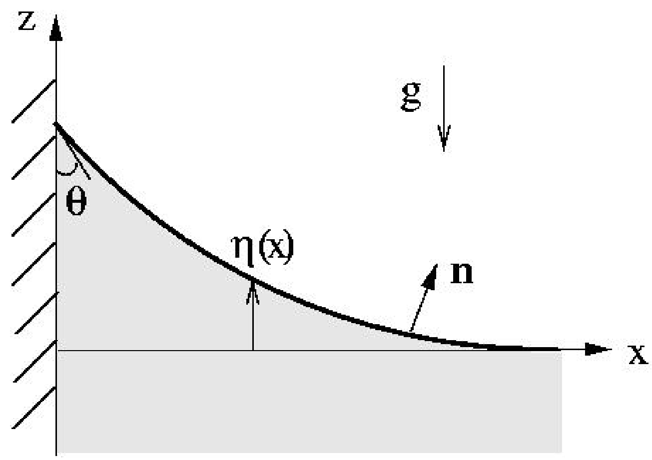

Figure 1: A definitional sketch of a planar meniscus at an air-water interface. The free surface is defined by $z = \eta(x)$, varying from its maximum elevation at its point of contact with the wall ($x = 0$) to zero at large x. The shape is prescribed by the Young-Laplace equation.

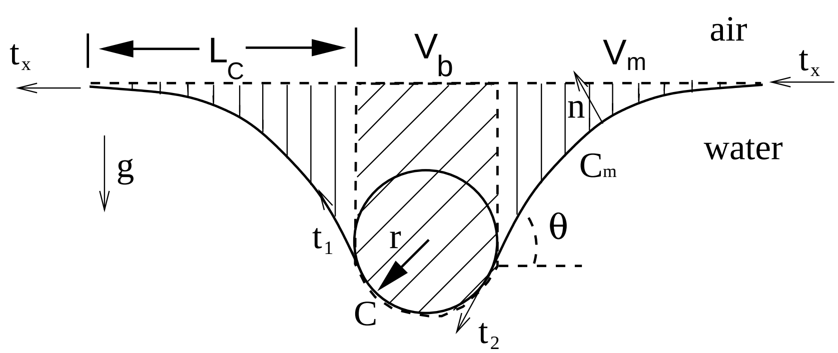

Figure 2: A non-wetting two-dimensional body of radius r and mass M floats on a free surface with surface tension σ. In general, its weight Mg must be supported by some combination of curvature and buoyancy forces. V_c and V_m denote the fluid volumes displaced, respectively, inside and outside the line of tangency.