PROBLEM SET: THE NOTION OF TOPOLOGICAL DEGREE, IAP 2021

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(1) Let $f : S^1 \to S^1$ be a smooth map. A point $y \in S^1$ is called a regular value of f if $f'(x) \neq 0$ for any $x \in S^1$ such that $f(x) = y$. Show that if $y \in S^1$ is a regular value, then the set of preimages $f^{-1}(y) = \{x \in S^1 : f(x) = y\}$ is finite. You may use the fact that any sequence of points $x_1, x_2, x_3 \dots$ in S^1 contains a subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \dots$ which converges to some limit point in S^1.

(2) Let $f : S^1 \to S^1$ be a smooth map. Suppose that $y \in S^1$ is a regular value. Prove that the topological degree of f is equal to the difference $n - m$, where n is the number of points $x \in S^1$ such that $f(x) = y$ and $f'(x) > 0$ and m is the number of points $x \in S^1$ such that $f(x) = y$ and $f'(x) < 0$. Hence the topological degree can be thought of as the number of preimages of any regular value, counted with sign.

(3) It is a theorem that regular values always exist. Using this result, show that given any two maps $f : S^1 \to S^1$ and $g : S^1 \to S^1$, the topological degree satisfies the multiplicative property $\deg(f \circ g) = \deg(f) \cdot \deg(g)$, where $f \circ g : S^1 \to S^1$ is the composition of f and g.