56 mph: breaking the commutative law by just a bit

David A. Vogan, Jr.

January 6, 2020

I talked about some interesting $2 \times 2$ complex matrices

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. $$

Here are rules for multiplying these matrices:

$$A^2 = B^2 = C^2 = -I_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$AB = C = -BA, \quad BC = A = -CB, \quad CA = B = -AC.$$

Problems

1. Find $2 \times 2$ invertible complex matrices $X$ and $Y$ so that

$$XY = -YX.$$ 

Can you find $3 \times 3$ matrices with this property?

2. Suppose $z$ is a complex number not equal to 1 (think of $z$ as close to 1). Can you find $n \times n$ invertible complex matrices $U$ and $V$ with the property that

$$UV = VU \cdot (zI_n)?$$

Here $I_n$ is the $n \times n$ identity matrix. This is a math version of the Heisenberg “canonical commutation relations;” says $U$ and $V$ almost commute, but not quite.

3. Suppose $w$ is a complex number not equal to 0 (think of $w$ as close to 0). Can you find $n \times n$ complex matrices $P$ and $Q$ with the property that

$$PQ = QP + wI_n?$$

If $w$ is Planck’s constant, this is the canonical commutation relations: in a slightly different way, says that $P$ and $Q$ almost commute, but not quite.

4. Can you get different answers to (2) and (3) if you replace $\mathbb{C}$ by another field?