56 mph: breaking the commutative law by just a bit

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I talked about some interesting 2×2 complex matrices

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

Here are rules for multiplying these matrices:

$$A^2 = B^2 = C^2 = -I_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$AB = C = -BA, \qquad BC = A = -CB, \qquad CA = B = -AC.$$

Problems

1. Find 2×2 invertible complex matrices X and Y so that

   $$XY = -YX.$$

   Can you find 3×3 matrices with this property?

2. Suppose z is a complex number not equal to 1 (think of z as *close to* 1). Can you find $n \times n$ invertible complex matrices U and V with the property that

   $$UV = VU \cdot (zI_n)?$$

   Here I_n is the $n \times n$ identity matrix. This is a math version of the Heisenberg "canonical commutation relations;" says U and V almost commute, but not quite.

3. Suppose w is a complex number not equal to 0 (think of w as *close to* 0). Can you find $n \times n$ complex matrices P and Q with the property that

   $$PQ = QP + wI_n?$$

   If w is Planck's constant, this *is* the canonical commutation relations: in a slightly different way, says that P and Q almost commute, but not quite.

4. Can you get different answers to (2) and (3) if you replace $\mathbb{C}$ by another field?