56 mph: breaking the commutative law by just a bit

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I talked about some interesting 2×2 complex matrices

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

Here are rules for multiplying these matrices:

$$A^2 = B^2 = C^2 = -I_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$AB = C = -BA, \qquad BC = A = -CB, \qquad CA = B = -AC.$$

Problem solutions

1. **Find 2×2 invertible complex matrices X and Y so that**

$$XY = -YX.$$

Can you find 3×3 matrices with this property?

The matrices A and B above will do. For $n \times n$ matrices, if you take the determinant of the equation $XY = YX \cdot (zI_n)$, you get

$$\det(X)\det(Y) = \det(Y)\det(X) \cdot z^n.$$

If X and Y are invertible (so their determinants are not zero), this forces $z^n = 1$. So $z = -1$ and $n = 3$ is not possible.

2. **Suppose z is a complex number not equal to 1 (think of z as *close to* 1). Can you find $n \times n$ invertible complex matrices U and V with the property that**

$$UV = VU \cdot (zI_n)?$$

Here I_n is the $n \times n$ identity matrix. This is a math version of the Heisenberg "canonical commutation relations;" says U and V almost commute, but not quite.

As explained in the first solution, this is only possible if z is an nth root of 1; that is, $z = \exp(2\pi ik/n)$ for some integer k between 1 and $n-1$. (The case $k = n$ is not allowed because we're assuming $z \neq 1$.) For $k = 1$, one way to achieve this is

$$U = \begin{pmatrix} \exp(2\pi i/n) & 0 & 0 & \cdots & 0 \\ 0 & \exp(4\pi i/n) & 0 & \cdots & 0 \\ 0 & 0 & \exp(6\pi i/n) & \cdots & 0 \\ & & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots & \exp(2\pi in/n \end{pmatrix}$$

$$V = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \vdots & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

That is, U multiplies the kth coordinate by $\exp(2\pi ik/n$, and V sends the kth coordinate to the $(k+1)$st (and the nth to the first).

3. **Suppose w is a complex number not equal to 0 (think of w as *close to* 0). Can you find $n \times n$ complex matrices P and Q with the property that**

$$PQ = QP + wI_n?$$

If w is Planck's constant, this *is* the canonical commutation relations: in a slightly different way, says that P and Q almost commute, but not quite.

Taking the trace of the desired equation, and using the fact that $\mathrm{tr}(PQ) = \mathrm{tr}(QP)$, we get

$$\mathrm{tr}(PQ) = \mathrm{tr}(QP) + nw, \qquad 0 = nw,$$

and therefore $w = 0$, contradicting our hypothesis. So no solution is possible. (Physicists find solutions to the canonical commutation relations by using *infinite* matrices.)

4. **Can you get different answers to (2) and (3) if you replace $\mathbb{C}$ by another field?**

Over any field at all, you are still led to the equation $nw = 0$ in the field, and you want $w \neq 0$. This is only possible *if the field has finite characteristic dividing n*. In that case it *is* always possible. Simplest example is $n = 2$; in a field of characteristic 2 1 is equal to -1, so

$$P = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

leads to

$$PQ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad QP = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} = PQ - I.$$