I talked about some interesting $2 \times 2$ complex matrices

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$  

Here are rules for multiplying these matrices:

$$A^2 = B^2 = C^2 = -I_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$AB = C = -BA, \quad BC = A = -CB, \quad CA = B = -AC.$$  

Problem solutions

1. **Find $2 \times 2$ invertible complex matrices $X$ and $Y$ so that**

$$XY = -YX.$$  

**Can you find $3 \times 3$ matrices with this property?**

The matrices $A$ and $B$ above will do. For $n \times n$ matrices, if you take the determinant of the equation $XY = YX \cdot (zI_n)$, you get

$$\det(X) \det(Y) = \det(Y) \det(X) \cdot z^n.$$  

If $X$ and $Y$ are invertible (so their determinants are not zero), this forces $z^n = 1$. So $z = -1$ and $n = 3$ is not possible.

2. **Suppose $z$ is a complex number not equal to 1 (think of $z$ as close to 1). Can you find $n \times n$ invertible complex matrices $U$ and $V$ with the property that**

$$UV = VU \cdot (z I_n)?$$  

*Here $I_n$ is the $n \times n$ identity matrix. This is a math version of the Heisenberg “canonical commutation relations;” says $U$ and $V$ almost commute, but not quite.*
As explained in the first solution, this is only possible if \( z \) is an \( n \)th root of 1; that is, \( z = \exp(2\pi ik/n) \) for some integer \( k \) between 1 and \( n - 1 \). (The case \( k = n \) is not allowed because we’re assuming \( z \neq 1 \).) For \( k = 1 \), one way to achieve this is

\[
U = \begin{pmatrix}
\exp(2\pi i/n) & 0 & 0 & \cdots & 0 \\
0 & \exp(4\pi i/n) & 0 & \cdots & 0 \\
0 & 0 & \exp(6\pi i/n) & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & \exp(2\pi in/n)
\end{pmatrix}
\]

\[
V = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 1 \\
1 & 0 & 0 & \cdots & 0
\end{pmatrix}
\]

That is, \( U \) multiplies the \( k \)th coordinate by \( \exp(2\pi ik/n) \), and \( V \) sends the \( k \)th coordinate to the \((k+1)\)st (and the \( n \)th to the first).

3. Suppose \( w \) is a complex number not equal to 0 (think of \( w \) as close to 0). Can you find \( n \times n \) complex matrices \( P \) and \( Q \) with the property that

\[
PQ =QP + wI_n?
\]

If \( w \) is Planck’s constant, this is the canonical commutation relations: in a slightly different way, says that \( P \) and \( Q \) almost commute, but not quite.

Taking the trace of the desired equation, and using the fact that \( \text{tr}(PQ) = \text{tr}(QP) \), we get

\[
\text{tr}(PQ) = \text{tr}(QP) + nw, \quad 0 = nw,
\]

and therefore \( w = 0 \), contradicting our hypothesis. So no solution is possible. (Physicists find solutions to the canonical commutation relations by using infinite matrices.)

4. Can you get different answers to (2) and (3) if you replace \( \mathbb{C} \) by another field?

Over any field at all, you are still led to the equation \( nw = 0 \) in the field, and you want \( w \neq 0 \). This is only possible if the field has finite characteristic dividing \( n \). In that case it is always possible. Simplest example is \( n = 2 \); in a field of characteristic 2 1 is equal to \(-1\), so

\[
P = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\]

leads to

\[
PQ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad QP = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} = PQ - I.
\]