

Knots and Numbers:
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A little history.

It all began with smoke rings. You will recall that Gandalf could make smoke rings chase each other around and end up clustered above his head. Long before Tolkien, in 1858 to be precise, the renowned German mathematical physicist Hermann von Helmholtz (1821–1894) worked out the theory of “vortex rings,” and explained why they persisted in time. The smoke rotates around a circle in the core of the ring. Helmholtz showed that this dynamical system offers great resistance to getting broken.

This paper was read by the young Scottish physicist Peter Guthrie Tait (1831–1901). He was a school chum of James Clark Maxwell (and got a professorship at Edinburgh in 1859 in preference to Maxwell). In January, 1867, he succeeded in building an experimental apparatus that demonstrated Helmholtz’s claims about the persistence of vortex rings. They rotate through each other, as Gandalf’s did, one becoming small and squeezing through the other, only to grow and have the other squeeze through it, leapfrog style. They bounced off each other and vibrated. They acted like particles.

Tait was no slouch. He had just finished writing a textbook with William Thomson. Commoners were obliged to call Thomson Lord Kelvin now, because in 1866 he had been knighted for his role in the design of the first transatlantic telegraph cable; and of course he is known today also for his absolute scale of temperature. The book “T & T” set out the concept of energy and its conservation, clearly, for the first time. “The work was epoch-making, and created a revolution in scientific development,” wrote one commentator.

Thomson was in the audience when Tait demonstrated his apparatus. He was fascinated by Tait’s demonstration of Helmholtz’s ideas, and saw in these smoke rings the makings of a theory of the atom. Thomson’s vision was that perhaps atoms consisted of tiny vortex rings, persistent topological singularities in the aether. Different elements might correspond to different knot types!—so perhaps the hydrogen atom was a simple loop (the “unknot”), helium was the trefoil knot, (Here was something even Gandalf or the best modern vape artists couldn’t do.) These rings vibrated, putting them in different energy states. And perhaps the mystery of the formation of molecules could be explained by the linking of several knots to form links.

Now, you may think that Thomson was smoking something himself ... and his vision might be explained by the fact that the smoke in Tait’s apparatus was created by combining ammonia and sulfuric acid. But you have to remember that Rutherford’s image of a tiny nucleus surrounded by orbiting electrons appeared only in 1909, 42 years later; Ernest Rutherford wasn’t even born till 1871.

Kelvin’s idea became the subject of scorn and ridicule when it failed to account successfully for any facts of chemistry whatsoever. But I submit that the basic idea is very deep, and is the same as the idea underlying the current use of loops in a theory of

elementary particles, string theory: knots give a model for extended topological structures, and the combinatorial nature of the topology is what accounts for their persistence. Thomson's proposal can be thought of as reflecting a very early insight into the geometric notions underlying quantum mechanics. Today's ideas of string theory share this underlying idea, and it has re-entered chemistry in the study of polymer entanglement!

Some years later, starting in 1876, Tait began a serious attempt to list different knot types. Let's be clearer about what a knot is. Mathematically, a "knot" is a closed loop sitting in space in some way. Two knots are "the same" if one can be smoothly deformed to the other without ever crossing through itself. This operation is called a "regular isotopy." A "link" is a finite collection of nonintersecting embedded loops, so a knot is a link, and there is also the *empty* link.

Tait studied knots by drawing projections onto a plane. You can always arrange that this projection is free of kinks, tangencies, and points where three strands meet; this is then called a "knot projection." Tait published a correct and complete list of knots which can be presented using 7 or fewer crossings. The list on the back of your homework assignment was begun by him.

By the way, being Scottish, Tait was interested in golf, and in fact wrote a definitive account of the physics of golf. In most of the world he is known as the father of Freddie Tait, two time winner of the Scottish Open Golf Championship.

Some further notes on history:

The term "topology" (or rather its German equivalent "Topologie") was dreamed up in 1836 by Johann Benedict Listing (1808–1882). Listing was a student of Gauss, and besides his scientific talents he was something of a logodædalist. He is responsible also for the word "micron." He adopted many of Gauss's interests—in geodesy, magnetism, and in what Gauss called "Geometria Situs," or "the geometry of position." Gauss was following the title of a paper published in 1736 by Leonard Euler (1707–1783). Euler was, as you recall, the most prolific mathematician of all time; much of the approach taken in today's calculus courses stems from his writings, for example. In this paper he described the "Königsberg bridge problem": Is it possible to tour the city in such a way as to cross every bridge exactly once? This was the founding question of the field of graph theory.

The term didn't achieve acceptance till much later, incidentally—Henri Poincaré, often cited as the father of the subject, still referred to "Analysis situs" at the turn of the century. The word "topology" didn't become standard until it was advertised by Solomon Lefschetz from his stronghold in Fine Hall at Princeton University, where much of the subject as we know it today was created. He wrote a book in 1930 called *Topology*, and the world soon followed suit.

Another good illustration of the idea of topology is given by the "Möbius band," whose properties were discovered and published by Listing independently and somewhat earlier than by August Ferdinand Möbius (1790–1868). Whatever we choose to call it, this is a band with a twist in it. This twist persists under deformation; it is permanent, a more basic feature of the shape than any measurement. The number of twists—zero or one—is in effect a quantum number; it is a discrete and rigid property of a flexible, stretchable object. This is a characteristic of the study of Algebraic Topology.

Knots.

Tait and his contemporaries had no idea how to actually prove that the knots they wrote down couldn't be deformed into each other, and it was only well into this century that enough algebraic topology was developed to address these questions.

A cat has just come into my life, and in consequence I have some very tangled bits of yarn around the house. I know that they were originally “unknots” (topologists’ yarn always comes in closed loops) but I’ll be damned if I can return it to that state now. How do I know, then that my evil neighbor didn’t slip in and substitute a knotted bit of blue yarn? If I suspect that someone did, how could I prove, in a court of law, or to my colleagues here at MIT, that this thing could NEVER be returned to a simple loop? You need some “invariant,” which takes on one value for the unknot and another for this one.

To say it differently, what if I give a trefoil of yarn to my cat for a few days and then simplify the resulting tangle. How do I KNOW that it won’t come out the unknot, or some other knot?

The trefoil and its mirror image present another problem. If we put a mirror behind our knot and look at the image of the knot in it, we see what might be a different knot. The shape is the same but the crossings are reversed. The mirror image of the trefoil is different from the original trefoil: the trefoil is “chiral,” in contrast with the figure 8 knot, which is “amphicheiral”—a good spelling bee word introduced in 1904 by William Thomson. Here’s a property of knots which is clear and distinct, but very hard to guess in advance. The mirror image of K is written \overline{K} .

Given two knots I can cut each and splice the open ends together. This is an example of a general geometric operation called “surgery,” and the result is the “sum” of the two knots. It’s not quite well-defined yet though: I need to decide which ends to splice. This is resolved by giving the knots an “orientation.” Once again, reversing the orientation might or might not result in an isotopic oriented knot (though the first examples of this were only given in 1964). 8_{17} is the first non-reversible knot, and the knots you get by tying two 8_{17} ’s together in the two possible ways are distinct. The reverse of K is written $-K$.

A knot is “prime” if it can’t be written as a nontrivial sum. (To make the analogy with numbers better we should speak of the “product” of knots rather than their “sum,” but it’s too late now.) A feature of numbers is preserved: an oriented knot is a sum of prime oriented knots in a unique way (up to order of course). So people (starting with Tait) make tables of prime knots. The table on the back of this handout lists all the prime knots of 8 or fewer crossings. It does not carry complete information, though; it does not indicate whether K is isotopic to $-K$, \overline{K} , or $-\overline{K}$, or all or non of them, and it does not display \overline{K} even if K is chiral.

The number of distinct prime knots goes up very rapidly with the number of crossings in its most efficient projection. In this tabulation again K is considered equivalent to $-K$, \overline{K} , and $-\overline{K}$.

Crossings	3	4	5	6	7	8	9	10	11	12	13	14	15
Prime knots	1	1	2	3	7	21	49	165	552	2176	9988	46872	253293

Knot projections and Reidemeister moves, coloration

If you look at a knot from the right point, you can make what you see – its projection – avoid cusps, tangencies, and triple points.

If you watch a deformation, you won't be able to avoid these three bad things, but you will be able to avoid triple tangencies, quadruple points, and other pathologies.

So two knot projections represent the same knot exactly when they differ by a sequence of “Reidemeister moves.” Knot invariants are constructed by assigning some algebraic object – a number or a polynomial, for example – to knot projections in such a way that the value is unchanged by Reidemeister moves.

A basic idea in knot theory is to assign an invariant to each knot projection and then check that it is invariant under Reidemeister moves.

For example:

Definition. A knot projection is 3-colorable if you can assign an element of $\{R, G, Y\}$ to each arc in such a way that (1) at every crossing all three incoming arcs are either all the same color or all different colors, and (2) all three colors are used.

Observe that the standard trefoil (either of them) is 3-colorable, but that the standard unknot is not.

Lemma. The property of being 3-colorable is preserved by Reidemeister moves.

Corollary. The trefoil is nontrivial.

Rational tangles.

Many mathematicians have tried their hand at the problem of classifying knots. Among them is John Conway, a great group theorist and combinatorialist. When I first met him he was at the University of Cambridge, but he has since migrated to Princeton. He proposed to first isolate a ball containing a simple part of a knot he wished to study. He demanded that the knot pierce the surface of the ball in exactly four spots. Fix those points and call them NE, SE, SW, and NW. What lies inside the ball is a “tangle.” Two tangles are equivalent if you can move one to the other through tangles (and without detaching them from the sphere). Tangles are quite easy to draw. The “zero tangle” has two horizontal lines. Perhaps it's called “zero” because the slope is zero. That makes two vertical lines “infinity.” “1” will be a line of slope 1 with one of slope -1 passing under it, and “ -1 ” is the other way around. There are some simple moves one can do to generate new tangles from old. One can *Twist*: pass SE over NE. One can *Twist-inverse*: pass NE under SE. And one can *Rotate* (in the positive direction, by 90°). Call these operations T, T^{-1} , and R . For example, here's a picture of $T^n = RT^n =$ is a barber pole.

Here's a little fact you can check directly: $(TR)^3 = I$. Students don't appreciate the letter “F” as much as they might: it's the first letter of the alphabet to look 8 different ways under the symmetries of a square. Let's label a tangle by F, to keep track how it is moved around.

We assigned a rational number to some tangles. How far can this go? Here's the theorem, due to some combination of Schubert, Conway, Kauffman, and Goldman. Write

Tang for the set of isotopy classes of tangles. Actually, we allow *links* here too, consisting of several non-intersecting components. We'll distinguish the tangles obtained from $=$ by our operations by calling them “rational tangles.” They have no closed loops.

Theorem. There is a function $Q : \text{Tang} \rightarrow \mathbf{C} \cup \{\infty\}$ such that

- $Q_{=} = 0$,
- $Q_{TF} = 1 + Q_F$,
- $Q_{RF} = -1/Q_F$,
- $Q_{F \sqcup C} = 0$ if C is a separated unknotted circle.

These conditions uniquely define Q , and Q establishes a bijection from rational tangles to $\mathbf{Q} \cup \{\infty\}$.

So T not only twists, it also *translates*, and R not only rotates, it also *reciprocates*. For example, $Q_{||} = \infty$, $Q_{T^n} = n$, and the value of the barber pole is $-1/n$.

Since $R^2 : q \mapsto -1/q \mapsto q$, one finds from this is that on rational tangles, $R^2 = I$. This is an extra symmetry exhibited by rational tangles. You'll verify it for homework.

Square dance

Let's “prove” the rational part of this theorem by a square dance!

Two books on knot theory:

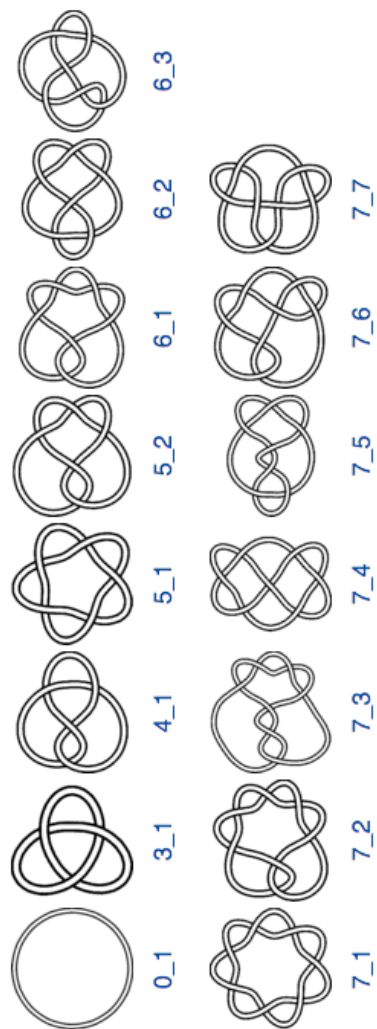
Colin Adams, *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots*, W. H. Freeman and Co., 1994.

Charles Livingston, *Knot Theory*, Math. Assoc. of America, 1993.

Continued fractions occur in any elementary number theory book, e.g.

Harold Davenport, *The Higher Arithmetic*, Dover, 1983.

Knots with 7 or fewer crossings



Knots with 8 crossings

