Discrete vs. Smooth Geometry

Discrete differential geometry attempts to construct a theory of geometry on piecewise-flat structures like polyhedra, while preserving structure from smooth differential geometry. A surprising property of this field is that there is "no free lunch:" Often times there is no single discrete theory that completely captures the smooth picture. We illustrate a simple example below.

Take $\gamma(t) : \mathbb{R} \to \mathbb{R}^2$ to be a curve in the plane. For this exercise, we will assume that $\|\gamma'(t)\|_2 \neq 0$ for all $t \in \mathbb{R}$ and that $\gamma(t)$ is a smooth function. Define the *arc length* function $s(t)$ as the function that goes from the parameter t of the curve to the arc length of the curve in interval $[0, t]$:

$$s(t) := \int_0^t \|\gamma'(\bar{t})\|_2 \, d\bar{t}.$$

Assume that $s(t)$ has a differentiable inverse function $t(s)$, so we can write $t(s(t_0)) = t_0$ and $s(t(s_0)) = s_0$.

Problem 1: Define $\gamma(s) := \gamma(t(s))$. Show that $\|\gamma'(s)\|_2 = 1$, where the prime indicates differentiation with respect to s.

We call the derivative $T(s) := \gamma'(s)$ the *unit tangent* of the curve $\gamma(s)$ at parameter s. If we think of a car driving along γ, by considering $\gamma(s)$ instead of $\gamma(t)$, the previous problem shows that our car drives with a *constant speed* of 1!

Problem 2: Argue that there exists a differentiable function $\theta(s)$ such that $T(s) = (\cos\theta(s), \sin\theta(s))$. Defining $N(s) := (-\sin\theta(s), \cos(\theta(s)))$ to be $T(s)$ rotated $90°$ in the plane, argue that $T'(s) = \kappa(s)N(s)$, where $\kappa(s) := \theta'(s)$. Draw a picture of a curve γ, and at a point on γ draw the corresponding vectors T and N.

Here, the vector $N(s)$ is the *normal vector* to the curve at $\gamma(s)$. The function $\kappa(s)$ is the *curvature* of γ. Intuitively, if we think of a car driving along γ with constant speed, then κN is the force experienced by the passengers as the driver turns the steering wheel to stay on γ; the angle of the steering wheel is given by θ'.

Problem 3: Suppose $\gamma(a) = \gamma(b)$ and that $T(a) = T(b)$; in other words, our car drove in a loop. Argue that $\theta(b) = \theta(a) + 2\pi k$ for some $k \in \mathbb{Z}$; conclude $\int_a^b \kappa(s)\, ds = 2\pi k$.

The quantity k is known as the *turning angle* between $s = a$ and $s = b$; it measures the number of times the car turned around in its path.

Now we're going to do something more advanced. Take another function $v(s) : [0, 1] \to \mathbb{R}^2$, and define a one-parameter family of curves $\gamma_r(s) := \gamma(s) + r \cdot v(s)$, where $r \in \mathbb{R}$; assume $v(0) = v(1) = (0, 0)$. We can think of the length of γ_r between $s = 0$ and $s = 1$ as a function

$$\ell(r) := \int_0^1 \|\gamma'(s) + r \cdot v'(s)\|_2 \, ds.$$

Problem 4 (optional): Show that $\ell'(0) = \int_0^1 \kappa(s) N(s) \cdot v(s)\, ds$.
Note: This problem is tricky! You'll need to apply differentiation under the integral sign and integration by parts, recalling our boundary conditions on $v(\cdot)$.

Now let's consider a *discrete curve*, consisting of a sequence of line segments between points $p_1, p_2, \ldots, p_n \in \mathbb{R}^2$.

Problem 5: Define θ_i to be the signed angle between the vector $p_i - p_{i-1}$ and the vector $p_{i+1} - p_i$. Argue that if $p_1 = p_n$, then $\sum_{i=1}^{n-1} \theta_i = 2\pi k$ for some $k \in \mathbb{Z}$.
Hint: Recall the exterior angle theorem from classical geometry.

The integer k is a discrete analog of the turning angle of our smooth curve γ! Based on this observation, we could use the angle θ_i to measure the curvature of our discrete curve at vertex p_i while preserving an analog of the turning angle formula from problem 3. Plus, θ_i is a reasonable measure of curvature: $\theta_i = 0$ when the two segments that meet at p_i are collinear.

We can think of the length of our discrete curve as a function

$$\ell(p_1, \ldots, p_n) := \sum_{i=1}^{n-1} \|p_{i+1} - p_i\|_2.$$

Problem 6 (optional): For $i \notin \{1, n\}$, show that $\|\nabla_{p_i} \ell(\cdot)\|_2 = \left|2 \sin \frac{\theta_i}{2}\right|$.
Note: This problem is also tricky! Draw lots of pictures and apply trigonometry liberally.

Here we have derived our "no free lunch" property! In the smooth picture, we derived two important places where the same curvature function $\kappa(s)$ appears: The turning angle theorem (problem 3) and the first variation of arc length (problem 4). If attempt to discretize curvature in such a way that the turning angle theorem is preserved, we likely would choose curvature at vertex p_i to be proportional to θ_i (problem 5). But, if we wish to discretize curvature so that it gives the proper derivative of arc length, we should use a value proportional to $2 \sin \frac{\theta_i}{2}$ (problem 6). Sadly we cannot have both at the same time!

At the same time, for $\theta \approx 0$, notice $2 \sin \frac{\theta}{2} \approx \theta$, so as we refine a discrete approximation of a smooth curve the two notions agree.