IAP Lecture: Calculus on Fractals, Winter 2019.
Homework

Some definitions and notations

Let $q_0 = (0,0)$, $q_1 = (1,0)$, and $q_2 = (1/2, \sqrt{3}/2)$. Define $F_i : \mathbb{R}^2 \to \mathbb{R}^2$ by $F_i(x) = \frac{1}{2}(x - q_i) + q_i$, $i = 0, 1, 2$.

The Sierpinski gasket is the unique nonempty compact subset SG of $\mathbb{R}^2$ such that

$$SG = \cup_{i=0}^{2} F_i(SG).$$

SG is the limit of graphs Γ_m with vertices in V_m where

$$V_0 = \{q_i\}_{i=0}^{2}, \qquad V_m = \bigcup_{i=0}^{2} F_i V_{m-1}.$$

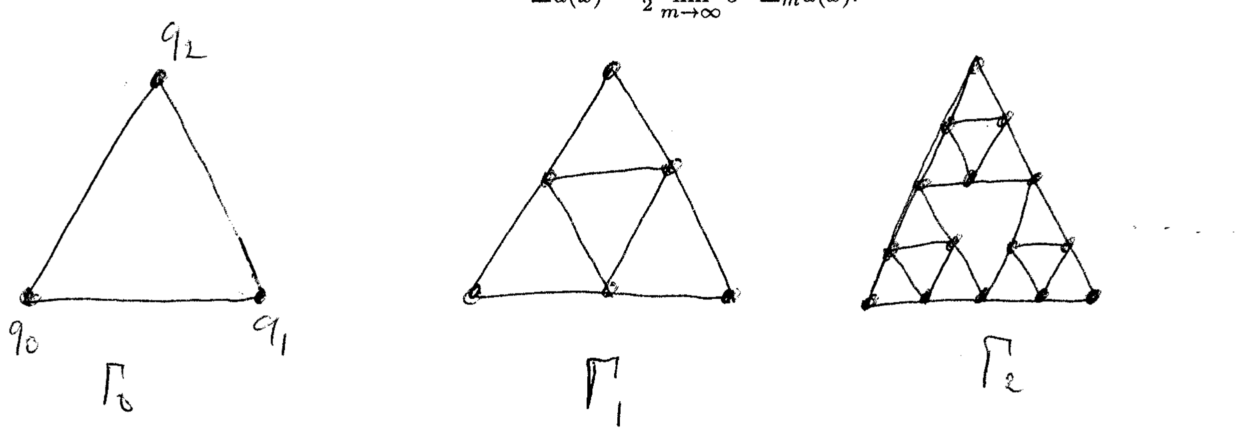
The graph Laplacian on Γ_m is defined by:

$$\Delta_m u(x) = \sum_{y \sim_m x} u(y) - 4u(x) \quad x \in V_m \setminus V_0,$$

where the notation $y \sim_m x$ means that the vertice y is a neighbor of x in the graph Γ_m.

Moreover, the Laplacian on SG is the operator defined by

$$\Delta u(x) = \frac{3}{2} \lim_{m \to \infty} 5^m \Delta_m u(x).$$



Problem 1. Let $q_0 = (0,0)$, $q_1 = (1,0)$, and $q_2 = (1/2, \sqrt{3}/2)$. Consider the restriction of a harmonic function h to the line segment in SG joining q_0 and q_1.

1.1. Find explicit formulas for $h(1/4)$ and $h(3/4)$ as a linear combination of $h(0), h(1/2), h(1)$.

1.2. Show that the previous algorithm localizes, so that the values of h at all dyadic points in the interval $(h(k2^{-n}),\ n \geq 2,\ k = 1, 2, \ldots, 2^n - 1)$ are determined by $h(0), h(1/2), h(1)$.

Problem 2.

2.1. Find all numbers λ and the corresponding non-zero vectors u such that $u(q_i) = 0$ for $i = 0, 1, 2$ and $-\Delta_1 u(x) = \lambda u(x)$ for $x \in V_1 \setminus V_0$. (Such a vector u is called a Dirichlet eigenvector of $-\Delta_1$ and the corresponding λ is a Dirichelt eigenvalue.)

2.2. Prove that each vector u found in part 2.1, extends uniquely to a Dirichlet eigenvector for $-\Delta_2$ on the graph Γ_2 and determine the corresponding eigenvalue. That is you need to find an extension $\tilde{u}$ of u such that $-\Delta_2 \tilde{u} = \tilde{\lambda} \tilde{u}$ on $V_2 \setminus V_0$ for some $\tilde{\lambda}$ that you must specify.

2.3. Using a dimension counting argument find all the (other) Dirichlet eigenvalues and eigenvectors of $-\Delta$.

2.4. Justify that you can generalize the above process to find an algorithm that compute the Dirichlet eigenvalues and eigenvectors of Δ_m for all $m \geq 3$.