1. Consider the difference equation
\[ H(x + 1, y + 1) - bH(x + 1, y) - bH(x, y + 1) + (2b - 1)H(x, y) = 0 \]
for an unknown function \( H(x, y) \) of \((x, y) \in \mathbb{Z}_0 \times \mathbb{Z}_0\) supplemented with boundary conditions
\[ H(x, 0) = \chi(x), \quad H(0, y) = \phi(y), \quad x, y \in \mathbb{Z}_0 \]
Where \( \chi \) and \( \phi \) are two given functions satisfying \( \chi(0) = \phi(0) \). Show that there exists a unique function \( H \) satisfying (1), (2).

2. Consider a partial differential equation
\[ H_{xy}(x, y) + \beta H_x(x, y) + \beta H_y(x, y) = 0, \quad x, y > 0, \]
for an unknown continuous function \( H(x, y) \) of \((x, y) \in \mathbb{R}_0 \times \mathbb{R}_0\) supplemented with boundary conditions
\[ H(x, 0) = \chi(x), \quad H(0, y) = \phi(y), \quad x, y \geq 0, \]
Where \( \chi \) and \( \phi \) are two given continuous functions satisfying \( \chi(0) = \phi(0) \).

A) Show that if \( H \) solves (3), then it also solves an integral equation
\[ H(X, Y) - \chi(0) - \phi(Y) + \chi(0) + \beta \int_0^X H(x, Y) dx + \beta \int_0^Y H(X, y) dy - \beta \int_0^X \chi(x) dx - \beta \int_0^Y \phi(y) dy = 0. \]

B) Show that there is at most one continuous function \( H(X, Y), X, Y \geq 0 \) solving (5).

Hint. Take two different solutions \( H \) and \( \tilde{H} \), consider \( \sup_{0 \leq x, y \leq \varepsilon} |H(x, y) - \tilde{H}(x, y)| \) for small \( \varepsilon > 0 \) and use (3) to find a contradiction.

3. Take two reals \( 0 < b_1, b_2 < 1 \) and consider a random walk on the lattice \( \mathbb{Z}_0 \times \mathbb{Z}_0 \), generalizing the \( b_1 = b_2 = b \) case studied in the lecture. Whenever the walker is travelling to the right, it continues right with probability \( b_1 \) and turns up with probability \( 1 - b_1 \). Whenever the walker is travelling up, it continues up with probability \( b_2 \) and turns to the right with probability \( b_1 \).

Recall that the height function of the walk is a 0/1 function defined at half-integers and which equals 1 whenever the path is below/to the right of a given point and 0 otherwise. In class we showed that for \( b_1 = b_2 = b \) case the expectation of the height function satisfies (1). Find a similar relation for the case \( b_1 \neq b_2 \).