18.095 HOMEWORK: PERSISTENT RANDOM WALKS AND TELEGRAPH EQUATION

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1. Consider the a difference equation

$$H(x+1,y+1) - bH(x+1,y) - bH(x,y+1) + (2b-1)H(x,y) = 0 \tag{1}$$

for an unknown function $H(x,y)$ of $(x,y) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ supplemented with boundary conditions

$$H(x,0) = \chi(x), \quad H(0,y) = \phi(y), \quad x, y \in \mathbb{Z}_{\geq 0} \tag{2}$$

Where χ and ϕ are two given functions satisfying $\chi(0) = \phi(0)$. Show that there exists a unique function H satisfying (1), (2).

2. Consider a partial differential equation

$$H_{xy}(x,y) + \beta H_x(x,y) + \beta H_y(x,y) = 0, \quad x, y > 0, \tag{3}$$

for an unknown continuous function $H(x,y)$ of $(x,y) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ supplemented with boundary conditions

$$H(x,0) = \chi(x), \quad H(0,y) = \phi(y), \quad x, y \geq 0, \tag{4}$$

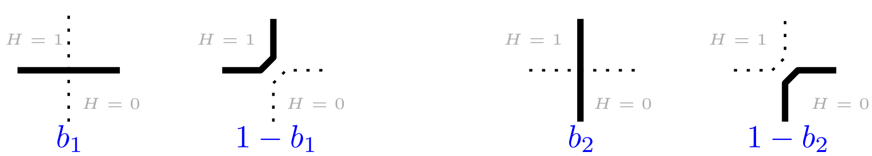
Where χ and ϕ are two given continuous functions satisfying $\chi(0) = \phi(0)$.
A) Show that if H solves (3), then it also solves an integral equation
(5)

$$H(X,Y) - \chi(X) - \phi(Y) + \chi(0) + \beta \int_0^X H(x,Y)dx + \beta \int_0^Y H(X,y)dy - \beta \int_0^X \chi(x)dx - \beta \int_0^Y \phi(y)dy = 0.$$

B) Show that there is at most one continuous function $H(X,Y)$, $X, Y \geq 0$ solving (5).

Hint. Take two different solutions H and $\tilde{H}$, consider $\sup_{0 \leq x,y \leq \varepsilon} |H(x,y) - \tilde{H}(x,y)|$ for small $\varepsilon > 0$ and use (5) to find a contradiction.

3. Take two reals $0 < b_1, b_2 < 1$ and consider a random walk on the lattice $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, generalizing the $b_1 = b_2 = b$ case studied in the lecture. Whenever the walker is travelling to the right, it continues right with probability b_1 and turns up with probability $1 - b_1$. Whenever the walker is travelling up, it continues up with probability b_2 and turns to the right with probability b_1.



Recall that the height function of the walk is a 0/1 function defined at half-integers and which equals 1 whenever the path is below/to the right of a given point and 0 otherwise. In class we showed that for $b_1 = b_2 = b$ case the expectation of the height function satisfies (1). Find a similar relation for the case $b_1 \neq b_2$.

Further related reading: A. Borodin, V. Gorin, A stochastic telegraph equation from the six-vertex model, 2018. arXiv:1803.09137