Ramsey theory

Jacob Fox

In the 1950's, Hungarian sociologist S. Szalai studied friendship relationships between children. He observed that in any group of around 20 children, he was able to find four children who were mutual friends, or four children such that no two of them were friends. Before drawing any sociological conclusions, Szalai consulted three eminent mathematicians in Hungary at that time: Erdős, Turán and Sós. A brief discussion revealed that indeed this is a mathematical phenomenon rather than a sociological one. For any symmetric relation R on at least 18 elements, there is a subset S of 4 elements such that R contains either all pairs in S or none of them. This fact is a special case of Ramsey's theorem proved in 1930, the foundation of Ramsey theory which developed later into a rich area of combinatorics.

The metastatement of Ramsey theory is that complete disorder is impossible. In other words, in a large system, however complicated, there is always a smaller subsystem which exhibits some sort of special structure.

More generally, we consider the following setting. We color the edges of K_n (a complete graph on n vertices) with a certain number of colors and we ask whether there is a complete subgraph (a clique) of a certain size such that all its edges have the same color. We shall see that this is always true for a sufficiently large n. Note that the question about frienships corresponds to a coloring of K_{20} with 2 colors, "friendly" and "unfriendly". Equivalently, we start with an arbitrary graph and we want to find either a clique or the complement of a clique, which is called an independent set. This leads to the definition of Ramsey numbers.

Ramsey number $R(s,t)$ is the minimum number n such that any red-blue coloring of the edges of K_n contains a red clique of order s or a blue clique of order t.

Theorem 1 (Ramsey 1930) *For all $s,t \in \mathbb{N}$, $R(s,t)$ exists.*

Theorem 2 *(***(Erdős-Szekeres 1935)** *For all $s,t \in \mathbb{N}$, $R(s,t) \leq R(s-1,t) + R(s,t-1)$.*

Proof: Let $n = R(s-1,t) + R(s,t-1)$. Consider a red-blue edge-coloring of K_n. Fix a vertex v. Consider two cases:

Case 1: There are at least $R(s-1,t)$ red edges incident with v. Then we apply induction on the red neighborhood of v, which implies that either they contain a red K_{s-1} or a blue K_t. In the first case, we can extend the red clique by adding v, and we are done.

Case 2: There are at least $R(s,t-1)$ blue edges incident with v. Then we apply induction on the neighborhood of v, which implies that either they contain a red K_s or a blue K_{t-1}. In the second case we can extend the blue clique by adding v, and we are done. $\square$

Using $R(s,2) = s$ and $R(2,t) = t$ and induction, we have the following corollary.

Corollary 1

$$R(s,t) \le \binom{s+t-2}{s-2}.$$

Diagonal Ramsey numbers $R(s,s)$ are of particular interest. The bound we have proved gives $R(s,s) \le \binom{2s-2}{s-1} \le 2^{2s}$.

This bound has not been improved significantly for over 75 years! All we know currently is that exponential growth is the right order of magnitude, but the base of the exponential is not known. The following is an old lower bound of Erdős. Note that to get a lower bound, we need to prove there is a 2-coloring such that there is no monochromatic clique of a certain size in either color. This is quite difficult to achieve by an explicit construction. (The early lower bounds on $R(s,s)$ were polynomial in s.)

The interesting feature of Erdős's proof is that he never presents a specific coloring. He simply proves that choosing a coloring at random almost always works! This was one of the first occurences of the probabilistic method in combinatorics. The probabilistic method has been used in combina- torics ever since with phenomenal success, using much more sophisticated tools; we will return to this later.

Theorem 3 *For $s \ge 3$, $R(s,s) \ge 2^{s/2}$.*

Proof: Let $n = 2^{s/2}$. Consider a random coloring of K_n where each edge is colored independently red or blue with probability $1/2$. For any particular s-tuple of vertices S, the probability that the clique on S has all edges of the same color is $2/2^{\binom{s}{2}}$. The number of s-tuples of vertices is $\binom{n}{s}$ and therefore the probability that there is at least one monochromatic s-clique is at most

$$\binom{n}{s} 2/2^{\binom{s}{2}} < \frac{n^s}{s!} 2^{1-\binom{s}{2}} = 2^{1+\frac{s}{2}}/s! < 1.$$

Therefore, with positive probability, there is no monochromatic clique of size s and such a coloring certifies that $R(s,s) > 2^{s/2}$. □

Determining Ramsey numbers exactly, even for small values of s, is a notoriously difficult task. The currently known diagonal values are: $R(2,2) = 2$, $R(3,3) = 6$, $R(4,4) = 18$. $R(5,5)$ is known to be somewhere between 43 and 49, and $R(6,6)$ between 102 and 165. A famous quote from Paul Erdős goes as follows: "Imagine an alien force, vastly more powerful than us, demanding the value of $R(5,5)$ or they will destroy our planet. In that case, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for $R(6,6)$. Then we should attempt to destroy the aliens."

In the early 1980s, Erdős, interested in the distribution of monochromatic cliques in edge-colorings, considered the following generalization of Ramsey's theorem. Suppose we color the edges of the complete graph on $[n] = 1,\ldots,n$, and each i is given a weight $w(i)$. The weight $w(S)$ of a subset $S \subset [n]$

is $\sum_{s\in S} w(s)$. How large of a weight monochromatic clique must one find? Denote this number by $f_w(n)$. The Erdős-Szekeres bound gives a monochromatic clique of order $\frac{\log n}{2}$. If $w(i) = 1$ for all i, then $f_w(n) \geq \frac{\log n}{2}$.

One of Erdős' favorite problems in combinatorics is the following. If $w(i) = 1/\log i$ for $i \geq 2$ (and $w(1) = 0$),

1. Does $f_w(n) \to \infty$?

2. Estimate $f_w(n)$.

Rödl answered the first problem, showing $f_w(n) \geq c\frac{\log\log\log\log n}{\log\log\log\log\log n}$.

Recently, Conlon, Fox, and Sudakov answered the second proved $f_w(n) = \Theta(\log\log\log n)$. The proof involves the combination of several tools from probabilistic and extremal combinatorics. But what about other choices for w? For which natural w does $f_w(n)$ convere/diverge? The next natural choice for the weight function is $w(i) = \frac{1}{\log i \log\log\log i}$. For this choice of weight function, $f_w(n) = \Theta(\log\log\log\log\log n)$. More generally, we have the following theorem. Here $\log_{(i)}(x)$ is the iterated logarithm given by $log_{(0)}(x) = x$ and, for $i \geq 1$, $\log_{(i)}(x) = \log(\log_{(i-1)}(x))$.

Theorem 4 *Let $w_s(i) = 1/\prod_{j=1}^{s} \log_{(2j-1)} i$. Then $f_{w_s}(n) = \Theta(\log_{(2s+1)} n)$. However, letting $w'_s = w_s(i)/(\log_{(2s-1)} i)$ for any fixed $\epsilon > 0$, then $f_{w'_s}(n)$ converges.*

Ramsey theory has had a considerable impact on number theory. In 1927, van der Waerden proved the following theorem.

Theorem 5 *For all positive integers k and r, there is another positive integer n such that every r-coloring of $[n]$ contains a monochromatic k-term arithmetic progression.*

The least such positive integer n is denoted by $W(k, r)$. Erdős and Turán in 1935 conjectured that a stronger result holds, that the largest color class must contain a k-term arithmetic progression. This was proved for $k = 3$ by Roth using Fourier analysis in the 1950's, and in general by by Szemerédi in the 1970's.

Theorem 6 *For ever k and $\epsilon > 0$, there is another positive integer n such that every subset $S \subset [n]$ with $|S| \geq \epsilon n$ contains a k-term arithmetic progression.*

The least such positive integer n is denoted by $S(k, \epsilon)$. We have $W(k, r) \leq S(k, 1/r)$.

There are now several different proofs of Szemerédi's theorem. One of the motivations for the Erdős-Turán conjecture was another conjecture, already several hundred years old. This conjecture states that the primes contain arbitrarily long arithmetic progressions. While the primes aren't dense (the prime number theorem says that the number of primes up to n is asymptotic to $n/\ln n$), good quantitative estimates for Szemeredi's theorem would imply this. However, the best known bound is due to Gowers:

$$S(k, \epsilon) < 2^{2^{\epsilon^{-2^{2^{k+9}}}}},$$

which is not strong enough to imply the conjecture on arithmetic progressions in primes. A few years ago, Green and Tao proved the conjecture.

Theorem 7 *The primes contain arbitrarily long arithmetic progressions.*

The proof extends Szemerédi's theorem from dense subsets of intervals to dense subsets of "pseudorandom" sets, and shows that the primes form a dense subset of such a pseudorandom set.

Another extension of van der Waerden's theorem was proved by Hales and Jewett in the 1960s. It is well known that the children's game Tic-Tac-Toe does not always have to have a winner. This is a game played in two dimensions on a $3x3$ board where players take turns filling in squares with X or O, trying to make a row, column, or diagonal all the same type. However, the Hales-Jewett theorem shows that in high enough dimensions, the analogue of Tic-Tac-Toe always has a winer. A combinatorial line is a set of k points $x_1, \ldots, x_k$ in $[k]^n$, which on each coordinate j, either it is constant on the k points, or the jth coordinate of x_i is j for $1 \leq j \leq k$.

Theorem 8 *For each k and r, there is an n such that every r-coloring of $[k]^n$ contains a monochromatic combinatorial line.*