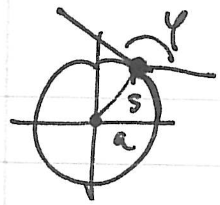
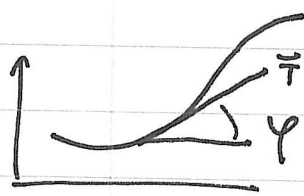


For a circle, $\varphi = \theta + \pi/2$

& $\theta = \frac{s}{a} \Rightarrow \varphi = \frac{s}{a} + \pi/2$

& $k = \frac{d}{ds}(\varphi) = \frac{1}{a}$.

In case we have curve given w/ y as a function of x,



$\varphi = \arctan\left(\frac{dy}{dx}\right)$

$\varphi = \arctan\left(\frac{dy}{dx}\right) = \arctan(y')$

so

$d\varphi = \frac{y''}{1+(y')^2}\,dx$. also, $ds = \sqrt{dx^2+dy^2}$

$= \sqrt{1+y'^2}\,dx$

so $\boxed{\frac{d\varphi}{ds} = \frac{y''}{(1+y'^2)^{3/2}} = k}$

& similar computations can be made in the general case.

E.g., parabola: $y = x^2$

Where is max curvature?

What about $y = x^4$?

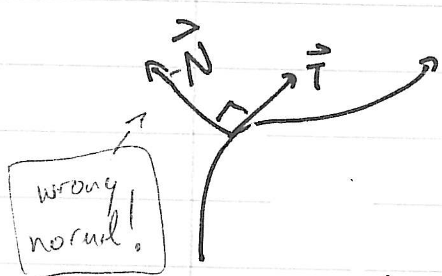
Observe that the unit tangent $\vec{T}$ is given by $\vec{T} = \langle \cos\varphi, \sin\varphi \rangle$, so

$$\frac{d\vec{T}}{d\varphi} = \langle -\sin\varphi, \cos\varphi \rangle,$$

a unit vector.

Moreover, $\vec{T} \cdot \frac{d\vec{T}}{d\varphi} = 0$, i.e., they are orthogonal ()

Thus, this vector is orthogonal to the curve; locally, $\vec{T}$ & $\vec{N} = \frac{d\vec{T}}{d\varphi}$ give us a nice new set of coordinates ; also, here



$$\frac{d\vec{T}}{ds} \overset{\text{chain rule}}{=} \frac{d\vec{T}}{d\varphi} \cdot \frac{d\varphi}{ds} = \vec{N} \cdot k$$

Why this "coordinate frame" is interesting?

Well, when we write

$$\vec{v} = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle \quad \& \quad \vec{a} = \left\langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} \right\rangle,$$

the coordinates don't separately tell us much about the physical behavior of the system but in terms of $\vec{T}$ & $\vec{N}$ nice things happen: already saw $\vec{v} = \vec{T} \cdot \frac{ds}{dt}$;

thus have $\vec{a} = \frac{d}{dt}\left(\vec{T} \cdot \frac{ds}{dt}\right)$

$$= \frac{d\vec{T}}{dt} \cdot \frac{ds}{dt} + \vec{T} \cdot \frac{d^2s}{dt^2}$$

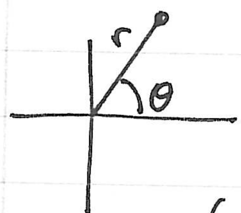
and $\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds} \cdot \frac{ds}{dt} = \vec{N} \cdot k \cdot \frac{ds}{dt}$

$$\vec{a} = \vec{N} \cdot \left(k \cdot \left(\frac{ds}{dt}\right)^2\right) + \vec{T} \cdot \frac{d^2s}{dt^2}$$

Ex: Find the curvature of $y = \sin x$. Where is it maximized/minimized?

Ex: Recall the cycloid w/ parametric eqns $\vec{R}(t) = r\langle t - \sin t, 1 - \cos t\rangle$. Compute $\vec{v}$, $\vec{a}$, & decompose $\vec{a}$ into tangential & normal components.

Polar coordinates : an alternate coordinate system, useful for describing certain curves or regions w/ circular symmetry. Idea: locate a point by distance from the origin (r) & the angle it makes w/ pos. x-axis (θ)



Obs: not unique representation (unlike rect. coords):

$$(r,\theta) = (-r, \theta+\pi) = (r, \theta+2\pi) = (-r, \theta-\pi) = \ldots$$

& also have $(0,\theta)$ = origin for every angle θ.

Given (r,θ), how to find (x,y)?

$x = r\cos\theta \quad y = r\sin\theta$

& how to go back? $r^2 = x^2 + y^2$, $\tan\theta = y/x$

The graph of an eqn in polar coords is the set of pts that satisfy the eqn

E.g: Graph
- $r = 1$
- $\theta = \pi/4$
- $r = 2\cos\theta$

Question: Which of the following points lie on the curve w/ eqn $r = \sin^2\theta$?
- $(2, \pi/4)$
- $(1, \pi/2)$ (tricky!)
- $(0, 3\pi/2)$