Vector functions & parametric equations

So far, have talked mostly about functions $y=f(x)$ where both x & y are real values, giving curves in the xy-plane.

(But not every curve is associated to a function, e.g.:)

Other ways to describe some of these curves: parametric equations

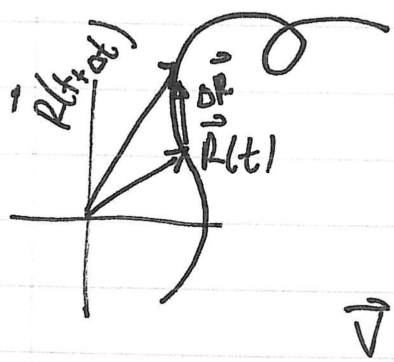
e.g. $x=\cos t$, $y=\sin t$ (3rd variable (parameter) t)

or equivalently as a vector-valued function

$$\vec{F}(t) = \langle \cos t, \sin t \rangle$$

In 3 space, same thing to describe motion of a particle: position $(x(t), y(t), z(t))$ a function of time, w/ position vector

$$\vec{R}(t) = \langle x(t), y(t), z(t) \rangle$$



The velocity vector of such a particle is

$$\vec{v}(t) = \lim_{t \to 0} \frac{\Delta \vec{R}}{\Delta t} =: \frac{d\vec{R}}{dt}$$

In coordinates, this is as nice as it could be:

$$\vec{v} = \frac{d\vec{R}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle = \langle x', y', z' \rangle = \langle \dot{x}, \dot{y}, \dot{z} \rangle$$

Newton & phys. notation

$$\text{Speed} = |\vec{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

$$= \frac{ds}{dt}, \quad \text{where } s \text{ is } \underline{\text{arclength}}$$

$$\left(\text{in 2D,} \quad \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}. \right)$$

Consequence: to compute arclength from time 0 to time T, have

$$s = \int ds = \int_0^T \frac{ds}{dt}\, dt = \int_0^T \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

<u>Note</u>: velocity is always tangent to curve (Law of Inertia), so

$$\hat{T} = \hat{v} = \frac{\vec{V}}{|\vec{v}|} \quad \text{is the unit tangent,}$$

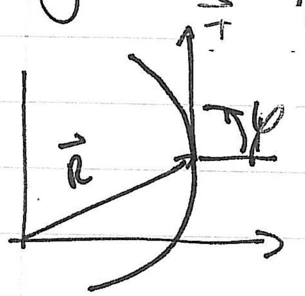
$$\text{and} \quad \vec{v} = \frac{ds}{dt} \cdot \vec{T}$$

$$\vec{a} = \text{Acceleration} = \text{change in velocity} = \frac{d\vec{v}}{dt}$$

$$= \frac{d^2\vec{R}}{dt^2} = \langle x'', y'', z'' \rangle = \langle \ddot{x}, \ddot{y}, \ddot{z} \rangle$$

Curvature and the unit normal

Consider (for the moment) a 2D curve given by $\vec{R}(t) = \langle x(t), y(t) \rangle$:



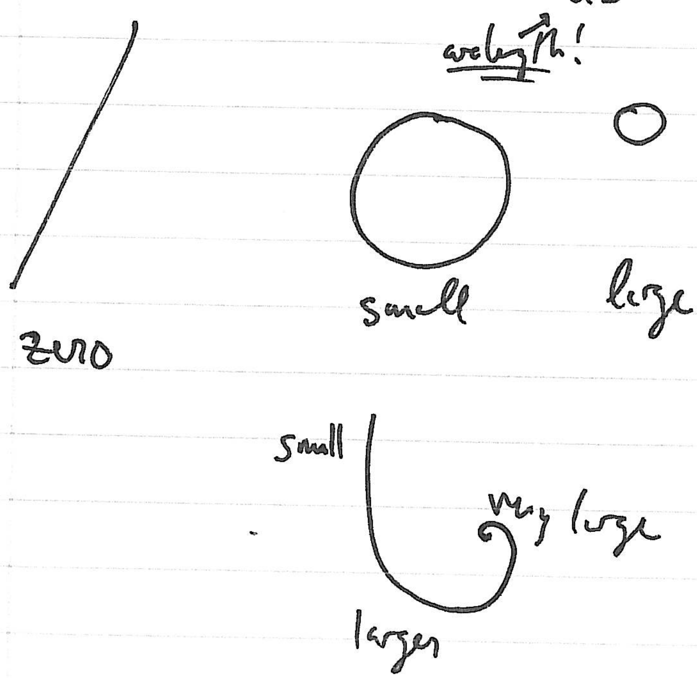
(also ϕ)

The angle φ between the unit tangent vector $\vec{T}$ & the horizontal changes as we move along the curve. The curvature is its rate of change

$$k = \frac{d\varphi}{ds}$$

arclength!

(which may be positive or negative, depending on which direction it curves.)



k : zero