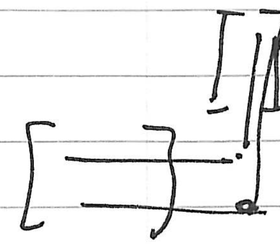
A matrix is just a rectangular (but we'll focus on square) table of #s. Significance:

they encode certain types of functions called linear transformations, in the following way:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$M\vec{v} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$



matrix acting on $\vec{v}$; like $M(\vec{v})$

So what? Well, the following operations are special linear transformations:

- projections (onto lines through $\vec{0}$)
- rotations (around $\vec{0}$)
- scalings (centered at $\vec{0}$)
- reflections (through lines through $\vec{0}$)

& in general, a function F is linear if and only if

$$F(c \cdot \vec{v}) = c \cdot F(\vec{v})$$

$$\& \quad F(\vec{v} + \vec{w}) = F(\vec{v}) + F(\vec{w})$$

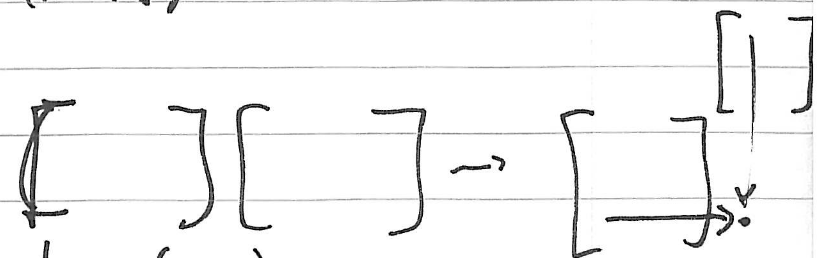
for all $\vec{v}, \vec{w}$.

(Motivation: very natural (aside), show up all sorts of places.)

Matrix multiplication: idea is that

$$M(N \cdot \vec{v}) = (M \cdot N) \cdot \vec{v}.$$

In practice,



Not commutative! (e.g.)

Special matrix: The $n \times n$ <u>identity matrix</u>:

$$I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$ has property $I_n \cdot \vec{v} = \vec{v}$.

a (square) matrix M is invertible iff $\exists N$ s.t. $M \cdot N = I$ (in which case also $N \cdot M = I$)

<u>Thm</u>: a square matrix M is invertible iff $\det M \neq 0$

<u>Why?</u> well, $\det(A \cdot B) = \det(A) \cdot \det(B)$,

also $\det(A) \cdot \det(A^{-1}) = \det(A \cdot A^{-1}) = \det(I) = 1$

so in fact $\det(A^{-1}) = \dfrac{1}{\det(A)}$

Given a square (say, 3x3) matrix A s.t. $\det(A) \neq 0$, there is a unique matrix A^{-1} s.t. $A \cdot A^{-1} = I_3 = A^{-1} \cdot A$.

Observe that $\det(A^{-1}) = \dfrac{1}{\det(A)}$

How to compute the inverse: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

<u>Step 1</u>: Compute $\det A$. If $\det A = 0$, give up (no inverse exists).

<u>Step 2</u>: Calculate the <u>minors</u>

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

where M_{ij} is what we get when we cover up row i & column j & take the determinant.

<u>Step 3</u>: Multiply by the checkerboard $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

- The result is the matrix of <u>cofactors</u>.

<u>Step 4</u>: Take the transpose. The result is the <u>adjoint</u> A^* of A.

<u>Step 5</u>: $A^{-1} = \dfrac{1}{\det A} \cdot A^*$

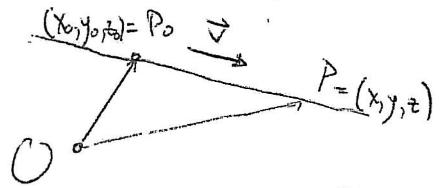
<u>Obs</u>: This works for matrices of <u>any</u> size.

E.g.: what's the inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$?

(Check that their dets are inverses.)

Lines & planes in 3-space.

A line is determined by a point P_0 & a direction (nonzero vector parallel to it) $\vec{v}$. Every point on a line



can be found by starting at P_0 & moving in direction $\vec{v}$ for some distance

$\Rightarrow$ There is some t, $\overrightarrow{P_0P} = t\vec{v} = t\cdot\langle a,b,c\rangle$

$$\langle x-x_0, y-y_0, z-z_0\rangle = \langle ta, tb, tc\rangle$$

So every point on the line is given by

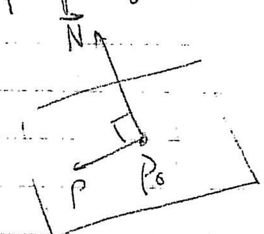
$$x = x_0 + ta$$
$$y = y_0 + tb$$
$$z = z_0 + tc$$

We can also eliminate t between these (parametric) equations to get

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c},$$

the symmetric equations for the line.

A plane is determined by a point P_0 & a <u>normal vector</u> $\vec{N}$ perpendicular to it. For any $P=(x,y,z)$ in plane,



$$\overrightarrow{P_0P} \perp \vec{N} \Rightarrow \vec{N}\cdot\overrightarrow{P_0P} = 0$$
$$\Rightarrow a(x-x_0)+b(y-y_0)+c(z-z_0)=0,$$
equiv
$$ax+by+cz=d$$

Also, a plane is determined by 3 noncolinear points, P_0, P_1, P_2. Get $\vec{N}$ how?

$$\vec{N} = \overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}.$$

Exercises: • Line through $(1,-1,0)$ & $(2,1,-2)$? Does this line contain $(3,3,4)$?
Plane through $(1,0,0)$, $(2,1,0)$ & $(-1,0,1)$? Intersection of this plane & line?