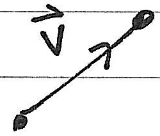
<u>Linear algebra</u>: vectors, matrices, determinants, etc.

<u>Vector</u>: geometrically, <u>distance</u> and a direction
(length; magnitude)

"How to get there from here"

$\vec{v}$ $Q = (q_1, q_2, q_3)$



$$\vec{v} = \overrightarrow{PQ} = \langle q_1 - p_1,\ q_2 - p_2,\ q_3 - p_3 \rangle$$

$P = (p_1, p_2, p_3)$

"Go $q_1 - p_1$ units in the x-direction, etc."

length $|\vec{v}| = |\overrightarrow{PQ}|$ = distance between P & Q

if $\vec{v} = \langle a, b, c\rangle$ then $|\vec{v}| = \sqrt{a^2 + b^2 + c^2}$

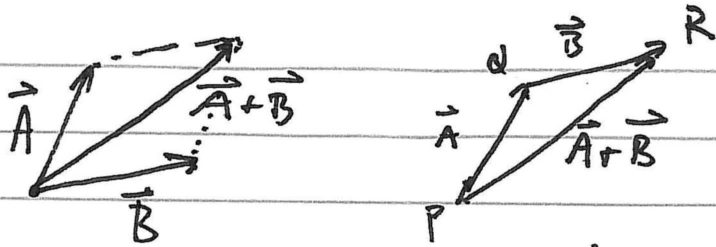
The zero vector is $\vec{0} = \langle 0,0,0\rangle$, the unique vector of length 0. Every other vector has direction given by a <u>unit vector</u> $\hat{u}$, i.e., $|\hat{u}| = 1$.

$$\operatorname{dir}(\vec{v}) = \frac{\vec{v}}{|\vec{v}|}.$$

N.B: two vectors are the same if they only differ "in location"

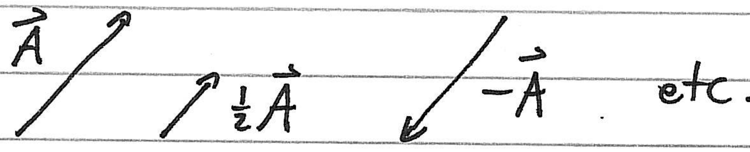
Can also work w/ vectors algebraically.

Can add them:



$(\vec{PR} = \vec{PQ} + \vec{QR})$

& can multiply a vector by a scaler (i.e., a real no.)



& can do these component-wise, i.e.,

$$\langle a_1, \ldots \rangle + \langle b_1, \ldots \rangle = \langle a_1 + b_1, \ldots \rangle$$

& $c \cdot \langle a_1, a_2, \ldots \rangle = \langle c a_1, c a_2, \ldots \rangle$

real number; not a vector. Scaler

Vector operations: Dot product. (2 or 3 (or more!) dimensions)

Given vectors $\vec{A}, \vec{B}$, define the dot product $\vec{A} \cdot \vec{B} = |A||B|\cos\theta$

(a.k.a. scaler product).

Scalar

(Why? One motivation is physics: constant force $\vec{F}$ moving an object a displacement $\vec{D}$ does work $\vec{F} \cdot \vec{D}$.)

Useful properties

- $\vec{A} \cdot \vec{A} = |\vec{A}|^2$

- $\vec{A}\cdot\vec{B} = 0 \iff \vec{A} \perp \vec{B}$
- $(c\vec{A})\cdot\vec{B} = c(\vec{A}\cdot\vec{B}) = \vec{A}\cdot(c\vec{B})$
- $\vec{A}(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$
- $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$

$\circledast$ if $\vec{A} = \langle a_1, a_2, a_3\rangle$ & $\vec{B} = \langle b_1, b_2, b_3\rangle$
Then $\vec{A}\cdot\vec{B} = a_1b_1 + a_2b_2 + a_3b_3$.

This last one makes dot products easy to compute from coordinates, leads to

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\cdot|\vec{B}|}$$ as a useful formula.

Special vectors: The <u>coordinate unit vectors</u> $\hat{i}, \hat{j}, \hat{k}$

3D: $\hat{i} = \langle 1,0,0\rangle$ $\hat{j} = \langle 0,1,0\rangle$ $\hat{k} = \langle 0,0,1\rangle$
2D: $\hat{i} = \langle 1,0\rangle$ $\hat{j} = \langle 0,1\rangle$

Observe that every vector can be written uniquely in terms of these vectors as

$$\vec{a} = \langle a_1, a_2, a_3\rangle = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

<u>Exercise</u>: for what value of z is $\langle 2,1,z\rangle \perp \langle 1,-3,2\rangle$?

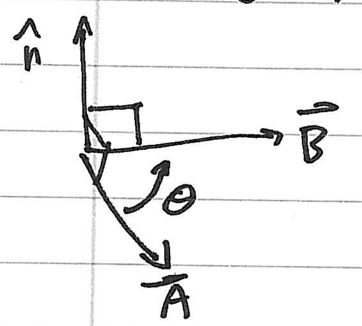
3D-only!

Cross products and determinants

A second product on vectors is the cross product or vector product

$\vec{A} \times \vec{B} = |\vec{A}| \cdot |\vec{B}| \cdot \sin\theta \cdot \hat{n}$, where

$\hat{n}$ is the unit normal to the plane containing $\vec{A}$ & $\vec{B}$, chosen by the right-hand rule, & θ is measured from $\vec{A}$ to $\vec{B}$.



(Physical motivation: a force $\vec{F}$ acting on a lever $\vec{r}$ creates torque $\vec{r} \times \vec{F}$)

<u>Obs</u>: $|\vec{v} \times \vec{w}| = |\vec{v}|\,|\vec{w}| \cdot |\sin\theta|$ is the area of the parallelogram w/ edges $\vec{v}$ & $\vec{w}$.

<u>Obs</u>: $\vec{v} \times \vec{w}$ is perpendicular to both $\vec{v}$ & $\vec{w}$.

<u>Obs</u>: $\vec{0} \times \vec{v} = \vec{v} \times \vec{0} = \vec{0}$ for all $\vec{v}$, & $\vec{v} \neq \vec{0}$, $\vec{w} \neq \vec{0}$, $\vec{v} \times \vec{w} = \vec{0} \Rightarrow \vec{v} \parallel \vec{w}$.

<u>Exercise</u>: $\hat{\imath} \times \hat{\jmath} = ?$ $\quad \hat{\jmath} \times \hat{\imath} = ?$

<u>Properties of cross product:</u>

Not commutative, not associative $\left(\begin{array}{l}\hat{i}\times(\hat{i}\times\hat{j}) \text{ vs} \\ (\hat{i}\times\hat{i})\times\hat{j}\end{array}\right)$

- $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$
- $(c\vec{a})\times\vec{b} = c(\vec{a}\times\vec{b}) = \vec{a}\times(c\vec{b})$
- $\left.\begin{array}{l}\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c} \\ (\vec{a}+\vec{b})\times\vec{c} = \vec{a}\times\vec{c}+\vec{b}\times\vec{c}\end{array}\right\}$ distributes over $+$

Can also give a component-based formula for

$\vec{A}\times\vec{B}$: starting w/ $\vec{A} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$
$\vec{B} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$,
apply rules above repeatedly to get

$$\langle a_1, a_2, a_3\rangle \times \langle b_1, b_2, b_3\rangle = \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1\rangle$$

There's an odd sort of symmetry here; we pause to give a useful way to think of cross products so as to remember this formula.

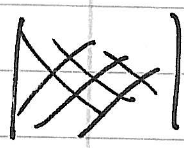
Determinants: to a square table of numbers, we associate a value, called the determinant, via the following recursive procedure:

$$\det([a]) = a$$

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a\cdot\det[d] - b\cdot\det[c]$$
$$= ad - bc$$

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = a\cdot\det\begin{bmatrix} e & f \\ h & i \end{bmatrix} - b\cdot\det\begin{bmatrix} d & f \\ g & i \end{bmatrix}$$
$$+ c\cdot\det\begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

etc.



$$= aei + bfg + cdh$$
$$- afh - bdi - ceg$$

but this doesn't work for 4x4 or larger!!!

Then we can write

$$\vec{A} \times \vec{B} = \det\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}.$$

Will often use short-hand

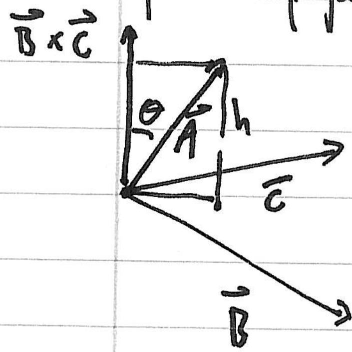
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \text{ for } \det\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Calculate $\langle 2,1,2\rangle \times \langle -1,-2,-1\rangle$ (or whatever)

Since we have determinants, may as well mention one further application: $\vec{A} = \langle a_1, a_2, a_3\rangle$, etc. Then

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \text{volume of}$$

parallelpiped (= box) spanned by $\vec{A}, \vec{B}, \vec{C}$:



Vol = base times height

base $= |\vec{B}\times\vec{C}|$

height = Proj of $\vec{A}$ onto $\vec{B}\times\vec{C}$

$$= |\vec{A}|\cdot\cos\theta$$

$$= \frac{\vec{A}\cdot(\vec{B}\times\vec{C})}{|\vec{B}\times\vec{C}|} \qquad \blacksquare$$

(& in general $|\text{Proj of } \vec{v} \text{ onto } \vec{w}| = \frac{\vec{v}\cdot\vec{w}}{|\vec{w}|}$)

A little more linear algebra (not in text?!): matrices, linear transformations, matrix multiplication