L'Hôpital's rule

Computing limits: some limits are easy, e.g.,

$$\lim_{x \to 3} \frac{\ln x}{x+4} = \frac{\ln 3}{7}$$

But some are not, e.g.,

$$\lim_{x \to 1} \frac{\ln x}{x-1} \longrightarrow \frac{0}{0}$$

$\uparrow$ $\boxed{\text{indeterminate form}}$

L'Hôpital's rule (L'Hop = student of J. Bernoulli)

says that if $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$ is indeterminate of

the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ & $\lim\limits_{x \to a} \frac{f'(x)}{g'(x)} = L$ exists

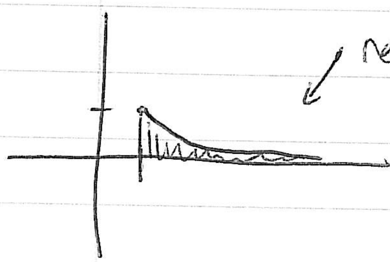
then $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = L$.

Ex: $\lim\limits_{x \to 0} \frac{1-\cos x}{x^2} = ?$

$$\lim_{x \to 0} \frac{\sin 4x}{2x+3} = ?$$

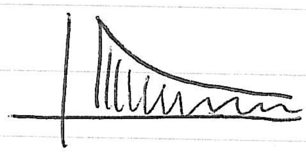
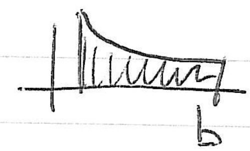
$$\lim_{x \to 1} \frac{\ln x}{x-1} = ?$$

Improper integration. What sense can one make of an integral like $\int_1^\infty \frac{1}{x^p}\,dx$?



region is unbounded, but under if area is infinite b/c is getting skinny so quickly.

Idea:

 $= \lim_{b\to\infty}$ 

So, is $\int_1^\infty \frac{dx}{x^p} := \lim_{b\to\infty} \int_1^b \frac{dx}{x^p}$ finite or infinite?

an improper integral … "infinitely wide."

If limit exists & is finite, say "converges." O/W, say "diverges." (maybe "diverges to infinity.")

In our case, $p \neq 1 \Rightarrow$

$$\int_1^{b} \frac{dx}{x^p} = \frac{x^{-p+1}}{-p+1}\Big]_1^b = \frac{b^{-p+1}}{-p+1} + \frac{1}{p-1}.$$

If $p<1$, $b^{-p+1} \xrightarrow{b\to\infty} \infty \Rightarrow$ diverges

$p>1$ $\quad b^{-p+1} \xrightarrow{b\to\infty} 0 \Rightarrow$ converges (to $\frac{1}{p-1}$)

If $p=1$, $\displaystyle\int_1^b \frac{dx}{x} = \ln x \Big]_1^b = \ln b \longrightarrow \infty$

$\Rightarrow$ diverges

Other examples:

$$\int_0^\infty e^{-x}\,dx = e^{-x}\Big]_0^\infty = 1 - \underbrace{e^{-\infty}}_{=0} = 1$$

$$\int_{-\infty}^\infty \frac{1}{x^2+1}\,dx = \arctan x \Big]_{-\infty}^\infty = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

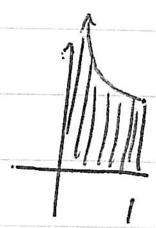
$$\int_e^\infty \frac{dx}{x\ln x} =$$

$$\int_e^\infty \frac{dx}{x(\ln x)^2} =$$

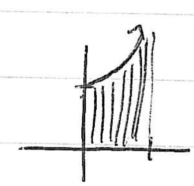
$$\int_0^\infty \cos x\,dx =$$

Can also consider regions that are "infinitely high"

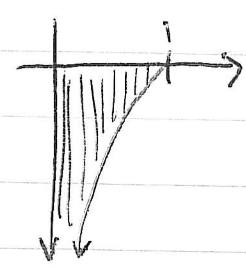
$$\int_0^1 \frac{1}{x^p}\,dx$$



or $$\int_0^1 \frac{dx}{\sqrt{1-x}}$$



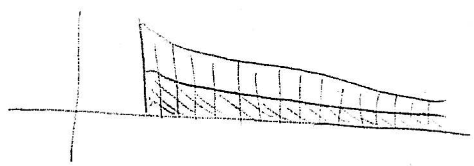
or $$\int_0^1 \ln x\,dx$$



When do improper integrals converge? Well, some we can integrate. Others, can use __comparison tests__: If $0 \le f(x) \le g(x)$ $\forall x$

Then $\int_a^\infty g(x)\,dx$ converges $\Rightarrow$ $\int_a^\infty f(x)\,dx$ converges

$\int_a^\infty f(x)\,dx$ diverges $\Rightarrow$ $\int_a^\infty g(x)\,dx$ diverges



__Limit comparison__. if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L \neq 0$ is finite

Then $\int_a^\infty f(x)\,dx$ has same behavior as $\int_a^\infty g(x)\,dx$

__E.g.__: $\int_1^\infty \frac{dx}{x^6 + 3x^3 - 2x^2 + 1}$

$\int_1^\infty \frac{\ln(x)}{x^3}\,dx$

__Infinite series__:

$\frac{1}{2} + \frac{1}{4} =$

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} =$

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} =$

$\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} =$

$\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \cdots =$

In general, if $S = 1 + \cancel{x} + x^2 + x^3 + \cdots + x^n$,

$x \neq 1$, then $(1-x)S = 1 + x + x^2 + \cdots + x^n$
$$- x - x^2 \cdots - x^n - x^{n+1}$$
$$= 1 - x^{n+1}$$

& so $S = \dfrac{1-x^{n+1}}{1-x}$.

If $|x| > 1$, $S \to \infty$ as $n \to \infty$,

so series diverges; if $|x| < 1$ then

(sum of infinitely many terms)

$$1 + x + x^2 + \cdots = \lim_{n\to\infty} S = \frac{1}{1-x}.$$

This is the geometric series.

Other examples of series: decimal expansions

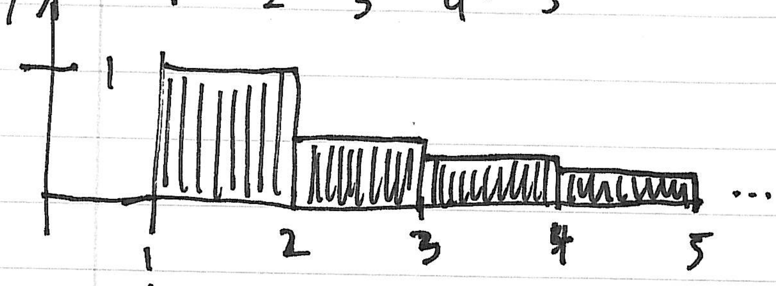
$\pi = 3.1415926\ldots$ means exactly that

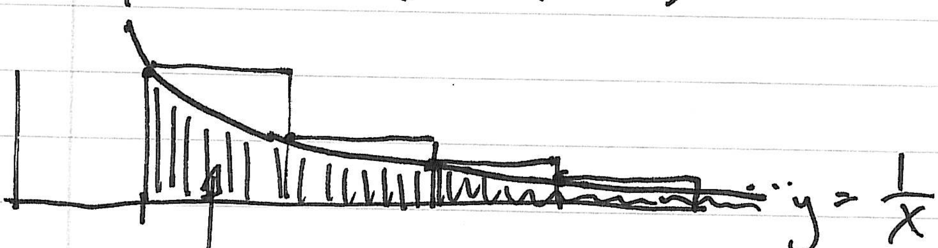
$$\pi = 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \cdots$$

If a series $a_0 + a_1 + a_2 + \dots$ converges, its terms **must** go to 0. But the
($\lim_{n\to\infty} a_n = 0$)

converse is $\boxed{\text{false}}$:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots =$$



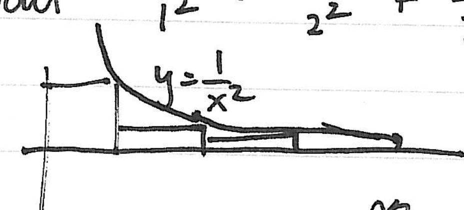


this area is $\int_1^\infty \frac{1}{x}\,dx = \infty$

so value of series is at least as big, is also infinite, i.e., series diverges.

What about $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$?

Well,



Area in boxes $< 1 + \int_1^\infty \frac{1}{x^2}\,dx = 2 \Rightarrow$ converges!

This gives us the integral test:

If $f(x) > 0$, $f'(x) \leq 0$, $\lim_{x \to \infty} f(x) = 0$

Then $\sum_{n=1}^{\infty} f(n)$ converges $\Longleftrightarrow$ $\int^{\infty} f(x)\,dx$ does

both directions!

What about other ways to tell if a series converges?

- Comparison test (just like for integrals)
- Ratio & Root tests

Ratio test:

Suppose $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = L$ exists, Then
$\boxed{a_n \geq 0}$

- $L < 1 \Rightarrow \sum_{n \geq 1} a_n$ converges
- $L > 1 \Rightarrow \sum_{n \geq 1} a_n$ diverges
- $L = 1 \Rightarrow$ test is inconclusive.

Idea: $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = L$ says a_n "looks like" $c \cdot L^n$, so converges when this geometric series converges.

Similar idea: <u>Root test</u>

$\boxed{a_n > 0}$ If $\lim_{n\to\infty} \sqrt[n]{a_n} = L$ then

- $L<1 \Rightarrow$ ~~co~~ $\sum a_n$ converges
- $L>1 \Rightarrow$ $\sum a_n$ diverges
- $L=1 \Rightarrow$ ~~th~~ inconclusive

E.g.: $\displaystyle\sum_{n\geq 1} \frac{1}{n!}$

$$\sum_{n} \frac{2^n}{n^n}$$

$$\sum_{n} \frac{n^3}{(\ln 2)^n}$$

$$\sum_{n} \frac{n!}{2^n}$$

So what? Well, main use is <u>Taylor series</u>.

<u>E.g.</u> $\frac{1}{1-x} = 1+x+x^2+\dots$

Integrate: $-\ln(1-x) = c + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$$

$$\arctan x = c + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots$$

$$\boxed{f(x) = a_0 + a_1 x + a_2 x^2 + \dots}$$

When happens? Well, for "nice enough" functions; can use ratio & root tests to check validity.

Important series: Those mentioned, plus

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

(Why for e^x? Well,

$$\frac{d}{dx}\left(a_0 + a_1 x + a_2 x^2 + \cdots\right) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

So need $a_0 = a_1$, $a_1 = 2a_2$, $a_2 = 3a_3$, ...

w/ $a_0 = e^0 = 1$)

& in general

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

Other series maybe of interest:

$$1+2x+3x^2+4x^3+\cdots \quad =$$

$$(1+x)^a = 1 + a\cdot x + \frac{a(a-1)}{2}x^2 + \frac{a(a-1)(a-2)}{2\cdot 3}x^3 + \cdots$$

Truncating these series give local approximations with polynomials,

e.g., $e^x \approx 1 + x + \frac{x^2}{2}$ when $x \approx 0$.

Similarly, $\sin x \approx x$ when $x \approx 0$ is a very good approx (2nd degree) often used in basic physics.