Now that we have some problems that integration solves, we can ask how to solve them: what sorts of things can we integrate?

Known methods:
- polynomials
- substitution or guessing (e.g., $\int v e^{v^2} dv$)
- some trig integrals (e.g., $\int \sin x^0 dx$, $\int \frac{1}{x^2+1} dx$)

New methods

1) Partial fractions: given a rational function (i.e., a ratio of two polynomials), there is a general method to integrate.

Eg.: $I = \int_3^4 \frac{3}{(x-1)(x-2)} dx = ?$

Method: $\frac{3}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$.

Solve for A, B (e.g., multiply out, then plug values or compare coefficients)

$3 = A(x-2) + B(x-1) \rightsquigarrow B = 3, A = -3$

$\Rightarrow I = \int_3^4 \frac{3}{x-2} - \frac{3}{x-1} dx = 3\ln(x-2) - 3\ln(x-1) \Big]_3^4$

$=$

General form: looks like $\dfrac{P(x)}{x(x+1)^2(x^2+1)} =$

$= \underbrace{\boxed{\text{Polynomial in } x}}_{\text{(long-divide)}} + \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2} + \dfrac{Dx+E}{x^2+1}$

(# coefficients = degree of denominator)

[2] Integration by parts

[See next page]

Problems: $\int x \sin x \, dx = ?$

$$\int_1^e x^a \cdot \ln(x)\, dx \qquad (a \neq -1)$$

[question: what if $a=-1$?]

Rule: $\int_a^b u\, dv = uv\Big]_a^b - \int_a^b v\, du$

Integration by parts

Recall the product rule: $\frac{d}{dx}\left(f(x)\cdot g(x)\right) = \left(\frac{d}{dx}f(x)\right)\cdot g(x) + f(x)\cdot\frac{d}{dx}(g(x))$

$$\Rightarrow \int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx$$

(sometimes written $\int u\,dv = uv - \int v\,du$:

$f(x) = v \Rightarrow dv = f'(x)dx$

$g(x) = u \Rightarrow du = g'(x)dx$)

Can use this to simplify many integrals:

$$\int x e^x\,dx = xe^x - \int e^x dx = xe^x - e^x$$

$u = x \quad dv = e^x dx$

$du = dx \quad v = e^x$

$$3\int x^2 \ln x\,dx = x^3\ln x - \int x^3\cdot\frac{1}{x}\,dx = x^3\ln x - \frac{x^3}{3}$$

$u = \ln x \quad dv = 3x^2 dx$

$du = \frac{1}{x}dx \quad v = x^3$

Informal rules:
- logarithms want to be differentiated
- polynomials want to be differentiated, but not as much as logs
- exponential & trig functions don't care.

Note: May have to integrate by parts more than once. Also,

$$I = \int e^x \sin x\,dx = -e^x\cos x + \int e^x\cos x\,dx = -e^x\cos x + e^x\sin x - \int e^x\sin x\,dx$$

$u = e^x \quad dv = \sin x\,dx$ $\qquad$ $u = e^x \quad dv = \cos x\,dx$

$$\Rightarrow I = \frac{e^x\sin x - e^x\cos x}{2}$$