Finishing up the Divergence Theorem:

<u>Thm</u>: $\iint_S \vec{F}\cdot\hat{n}\, dSA = \iiint_R \operatorname{div}\vec{F}\, dV$

Physical interpretation: flux out of region R whose boundary is surface S.

<u>Example problem</u>: Let R be the cylinder bounded by $z=0$, $z=b$, $x^2+y^2=a^2$. Compute the flux of the flow $\vec{F} = x\hat{i} - y\hat{j} + z\hat{k}$ out of R, first by computing surface integrals & second by applying the divergence theorem.

A few quick consequences:

1) Suppose $\vec{a}$ is a constant vector.
What is $\iint_S \vec{a}\cdot\hat{n}\, dSA$?

2) What is $\frac{1}{3}\iint_S \langle x,y,z\rangle\cdot\hat{n}\, dSA$?

3) What is $\iint_S \operatorname{curl}\vec{F} \cdot d\vec{S}A$?

__Back to curl__

__Recall:__ if $\vec{F}$ represents a fluid flow then

1) $\operatorname{dir}(\operatorname{curl}\vec{F})$ is the axis around which the fluid tends to rotate

2) $|\operatorname{curl}\vec{F}|$ measures the speed of this rotation

3) $\operatorname{curl}\vec{F} \cdot \hat{u}$ measures the speed of rotation around the axis $\hat{u}$ (where the sign gives direction of rotation according to the right-hand rule.

Given a curve C in space, the integral $\oint_C \vec{F} \cdot d\vec{R}$ measures the tendency of the fluid to flow along the curve C.

Given a (n orientable) surface S bounded by a closed curve C, we get a positive orientation if the surface is to the left of its normal vector as the normal walks along C.

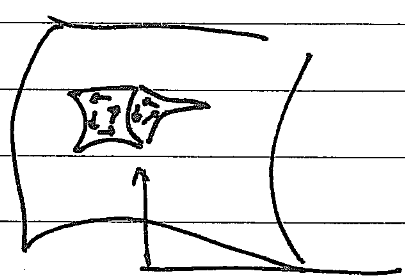
<u>Thm (Stokes' Thm)</u> Surface S w/ bounding curve C, field $\vec{F}$ [omitted niceness conditions] $\Rightarrow$

$$\oint_C \vec{F} \cdot d\vec{R} = \iint_S \operatorname{curl} \vec{F} \cdot \hat{n} \; dSA$$

or in other words

$$\oint_C \vec{F} \cdot \hat{T} \, ds = \iint_S \operatorname{curl} \vec{F} \cdot \hat{n} \, dSA$$

<u>Idea</u>: when surface is cut into patches,



cancellation except @ boundary.

Proof in one special case: suppose S is given by $z = g(x,y)$ for $(x,y) \in D \subseteq xy$-plane, boundary curve of D is C_1, & $\vec{F} = \langle M, N, P \rangle$.

Then $\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,dSA =$

$$= \iint_D \langle P_y - N_z,\ M_z - P_x,\ N_x - M_y\rangle \cdot \langle -g_x, -g_y, 1\rangle\, dA$$

$$= \iint_D -P_y g_x + N_z g_x - M_z g_y + P_x g_y + N_x - M_y \quad dA$$

While using parametrization $\overbrace{x = x(t),\ y = y(t)}^{C_1},\ z = g(x(t), y(t))$ (C)

$a \le t \le b$ we have

$$\int_C \vec{F}\cdot d\vec{R} = \int_a^b M\frac{dx}{dt} + N\frac{dy}{dt} + P\frac{dz}{dt}\,dt$$

$$= \int_a^b M\frac{dx}{dt} + N\frac{dy}{dt} + P\cdot\left(g_x\cdot\frac{dx}{dt} + g_y\cdot\frac{dy}{dt}\right)dt$$

$$= \int_a^b (M + Pg_x)\,dx + (N + Pg_y)\,dy$$

$$= \oint_{C_1} \qquad ''$$

Green's

$$= \iint_D \frac{\partial}{\partial x}(N + P\cdot g_y) - \frac{\partial}{\partial y}(M + P\cdot g_x) \quad dA$$

Now use chain rule, recalling that

$N = N(x, y, z)$ & $z = g(x, y)$.