Surface integrals

Surface given by $\vec{r}(\underbrace{u,v}_{\text{parameters}}) = \langle x(u,v), y(u,v), z(u,v) \rangle$

To compute a surface integral, cut surface up into many little pieces; on each piece, measure the contribution; then add up. "Pieces" are little Δu-by-Δv rectangles — contribution is $f(r(u,v))\, dA$

& recall $dA = |\vec{r_u} \times \vec{r_v}|\, du\, dv$

$\Rightarrow$ Surface integral is

$$\iint_S f(x,y,z)\, \underset{\substack{\uparrow \\ \text{surface} \\ \text{area}}}{dA} = \iint_D f(\vec{r}(u,v))\, |\vec{r_u} \times \vec{r_v}|\, \underset{\substack{\uparrow \\ \text{area in} \\ \text{u-v plane}}}{dA}$$

E.g. Compute $\iint_S x^2\, dSA$ where S is the sphere $x^2+y^2+z^2 = 2$.

As w/ other integrals, interest comes b/c we can measure things with them that have interest for other reasons, e.g., **Flux**: given a surface (which we think of as a netting) & a field $\vec{F}$ (fluid flow), what is the flux of fluid through the surface?

Ans: $\iint_S \vec{F} \cdot \hat{n} \, dSA$, where $\hat{n}$ is the unit normal $\hat{n} = \frac{r_u \times r_v}{|r_u \times r_v|}$.

Side note: orientable vs. non-orientable surfaces

orientable ← what we're interested in

non-orientable ← like Möbius strip

E.g.: What is the flux of the vector field $\langle x, y, z \rangle$ through the surface of the cylinder bounded by $z=0$, $z=b$ & $x^2+y^2=a$?

E.g. What is the flux of the vector field $\vec{F}(x,y,z) = z\hat{i} + y\hat{j} + x\hat{k}$ through the sphere $x^2+y^2+z^2=1$?

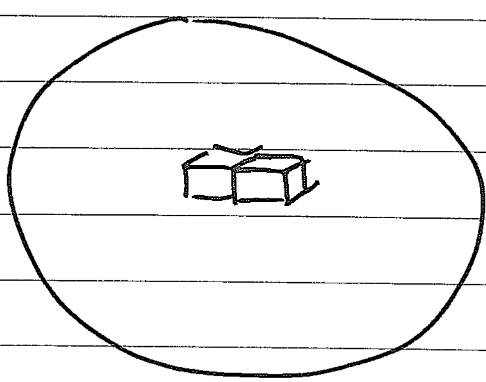
(Time-saver: $\vec{r}(\phi,\theta) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle$

$r_\phi \times r_\theta = \langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi \rangle$

$|r_\phi \times r_\theta| = \sin\phi$)

A surface is <u>closed</u> if it <u>encloses</u> a region (e.g., the sphere or the cylinder); in this case, we take $\hat{n}$ to be the <u>outwards</u> normal.

Now suppose we have a surface S enclosing a solid region R. Cut the region into many little boxes, & consider the field flowing through this region. The flux out of the entire region is $\iint_S \vec{F} \cdot \hat{n}\, dSA$. Also, the



flux is the sum over every little box of the flux out of the box (since flux on adjacent boxes cancel). For each of these boxes, the flux is (from yesterday) $\operatorname{div}\vec{F}\,\Delta V = \operatorname{div}\vec{F}\,dV$ in the limit

$\uparrow$ flux density

$$\Rightarrow \boxed{\iint_S \vec{F}\cdot\hat{n}\,dSA = \iiint_R \operatorname{div}\vec{F}\,dV}$$

Gauss' Thm / Divergence Theorem

<u>E.g.</u> Evaluate the same integrals from earlier using the divergence theorem