The operator ∇ ("del")

Recall the gradient $\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$ of a scaler function $f(x,y,z)$. We may think of this as the product of the vector $\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$ with the scaler f.

"vector differential operator"

With this notation, we can also combine ∇s in other ways: given a vector field $\vec{F}(x,y,z) = M\hat{i} + N\hat{j} + P\hat{k}$, have

$$\nabla \cdot \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \quad \text{is a scaler field}$$

&

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & P \end{vmatrix} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\hat{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\hat{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}.$$

divergence

curl

(Also have $\nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$, the Laplacian.)

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} \qquad \operatorname{curl} \vec{F} = \nabla \times \vec{F}$$

scaler field $\qquad$ vector field

E.g. Compute div $\vec{F}$ & curl $\vec{F}$ for

$$\vec{F}(x,y,z) = \langle 2x^2y,\ 3xz^3,\ xy^2z^2 \rangle$$

E.g. For $\vec{F}$ as above, compute $\nabla\cdot(\nabla\times\vec{F})$
$= \text{div}(\text{curl}\,\vec{F})$

(In fact, in general we have

$$\nabla\cdot(\nabla\times\vec{F}) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P_y - N_z,\ M_z - P_x,\ N_x - M_y \rangle$$
$$= (P_{yx} - N_{zx}) + (M_{zy} - P_{xy}) + (N_{xz} - M_{yz})$$
$$= 0.\ )$$

E.g. For $f(x,y,z) = xy + xz + y^2 + z^2$, compute $\text{curl}(\nabla f)$.

(In fact in general we have $\text{curl}(\nabla f) = \vec{0}$, & this completely captures the test for being a gradient field we mentioned yesterday.)

So we have a test "Is a vector field a gradient?" by taking curl & checking that it's $\vec{0}$ or not
↖ (& defined in a simply-connected region)

& we also have a test "Is a vector field a curl?" by taking div & checking whether it's 0 or not.

Aside: Green's Theorem may be restated in terms of "2D divergence" & "2D curl" – see the OCW notes.

$$\oint_C \vec{F} \cdot d\vec{R} = \iint_D (\operatorname{curl} \vec{F}) \cdot \hat{k} \, dA$$

(where we think of $\vec{F} = M\hat{i} + N\hat{j} + 0\hat{k}$ ← zero)

$$\& \quad \oint_C \vec{F} \cdot \hat{n} \, ds = \iint_D \operatorname{div} \vec{F} \, dA$$

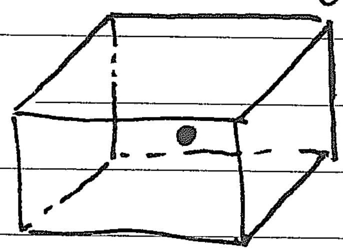
Okay, but what's the point?

Physical interpretations of div & curl

Divergence: want to know, is our fluid flowing into or out of a small region (e.g., locally is the fluid being compressed or expanding)?

To do this, we need to look "right around" the point P & see what the flow is doing:

take a small rectangular box w/ edges $\Delta x, \Delta y, \Delta z$ centered at your point (x,y,z).



What is flux through top face?

Well, it's $\langle 0,0,1\rangle \cdot \langle M(x,y,z) + \frac{1}{2}\Delta z M_z(x,y,z),\ N(x,y,z) + \frac{1}{2}\Delta z N_z(x,y,z),\ P(x,y,z) + \frac{1}{2}\Delta z P_z(x,y,z) \rangle$

$\cdot \Delta y \Delta z$

$$= P(x,y,z) + \tfrac{1}{2}\Delta x \Delta y \Delta z\ P_z(x,y,z).$$

Bottom face? $\langle 0,0,-1\rangle \cdot \langle \sim, \sim, P(x,y,z) + \frac{1}{2}(-\Delta z) P_z(x,y,z)\rangle$

$$= -P(x,y,z) + \tfrac{1}{2}\Delta x\Delta y\Delta z\, P_z(x,y,z)$$

Top & bottom together?	$\Delta x\Delta y\Delta z\, P_z(x,y,z)$
L & R	$\Delta x\Delta y\Delta z\, N_y(x,y,z)$
Front & back	" $M_x(x,y,z)$

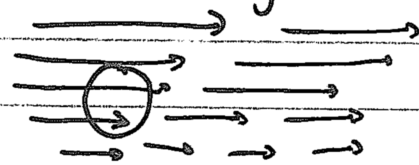
Total flux per unit volume? $M_x + N_y + P_z = \nabla \cdot \vec{F}(x,y,z)$

So div measures "compression" of a flow;

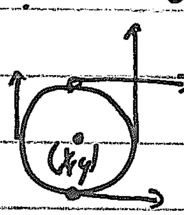
if $\operatorname{div}\vec{F} = 0$ everywhere we say $\vec{F}$ is <u>incompressible</u>;

if $\operatorname{div}\vec{F} > 0$, a <u>source</u>; if $\operatorname{div}\vec{F} < 0$, a <u>sink</u>.

Next, curl: imagine a small, rough sphere suspended in a flowing fluid



The fluid passing by may cause it to rotate. How much? Well, consider first just the 2-D case:



$-\left(M(x,y) + \frac{\Delta y}{2} M_y(x,y)\right)$

$M(x,y) - \frac{\Delta y}{2} M_y(x,y)$

$\Delta x = \Delta y = r$

$\rightsquigarrow r\left(N_x - M_y\right)$

(actually properly we should take a line integral; works out the same.)

A similar but trickier analysis can be done in 3D; the result is as follows:

- $\operatorname{dir}(\operatorname{curl}\vec{F})$ is the axis around which rotation happens
- $|\operatorname{curl}\vec{F}|$ measures angular velocity of the rotation

So if a fluid satisfies $\operatorname{curl}\vec{F} = \vec{0}$, it is "irrotational" — no eddies or whirlpools