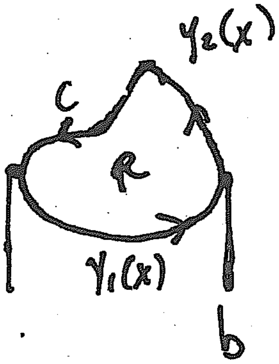


$$\oint_C M(x,y)\,dx = \int_a^b M(x,y_1(x))\,dx + \int_b^a M(x,y_2(x))\,dx$$

$$= \int_a^b M(x,y_1(x)) - M(x,y_2(x))\;dx$$

$$= -\int_a^b \left(M(x,y) \Big|_{y=y_1(x)}^{y_2(x)} \right) dx$$

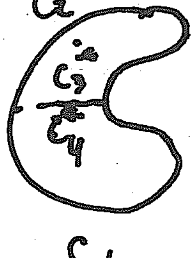
$$= -\int_a^b \int_{y_1(x)}^{y_2(x)} \frac{\partial M}{\partial y}\;dy\,dx$$

$$= -\iint_R \frac{\partial M}{\partial y}\;dA$$

& sim. $\displaystyle\oint_C N(x,y)\,dy = \iint_R \frac{\partial N}{\partial x}\,dA$

& get Green's Theorem by addition.

For other regions, cut them into simple pieces.



$M\,dx+N\,dy$

$$\oint_{C_1+C_2} = \oint_{C_1} + \oint_{C_3} + \oint_{-C_3} + \oint_{C_2}$$

$$= \oint_{C_1+C_3} + \oint_{C_2-C_3} = \iint_{R_1} \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dA$$

$$+ \iint_{R_2} \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dA =$$

$$= \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dA \quad \checkmark$$

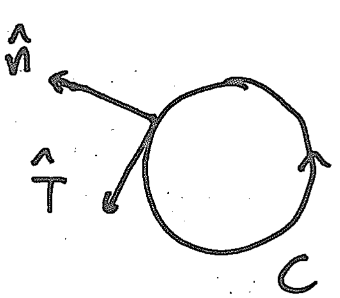
<u>Aside</u>: Tangential vs normal form of Green's Thm.

We have $\oint_C \vec{F}\cdot d\vec{R} = \oint_C \vec{F}\cdot\hat{T}\,ds$ is the integral of the tangential component of $\vec{F}$ around C, i.e., in physical terms it is the work done by a field. We can instead consider

$\oint_C \vec{F}\cdot\hat{n}\,ds$, the integral of the normal component of $\vec{F}$ on C. Physical interpretation: $\vec{F}$ is a flow, $\oint_C \vec{F}\cdot\hat{n}\,ds$ is total flow out of region bounded by C (where $\hat{n}$ is the outward-pointing normal). We can rewrite this:

$$\hat{n} = \text{Rot}(\hat{T}\ 90^\circ \circlearrowright) = \left\langle \frac{dy}{ds}, -\frac{dx}{ds}\right\rangle$$

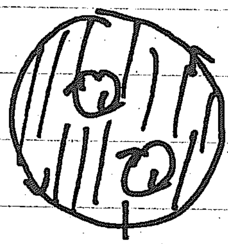
$$\hat{T} = \left\langle \frac{dx}{ds}, \frac{dy}{ds}\right\rangle$$



$$\Rightarrow \oint_C \vec{F}\cdot\hat{n}\,ds = \oint_C M\,dy - N\,dx$$

& can evaluate using Green's.

What about Green's Theorem on regions that do have holes? Just positively orient the entire boundary & add up the different pieces.



$$\iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dA = \oint_{C_1} M\,dx + N\,dy + \oint_{C_2} M\,dx + N\,dy + \oint_{C_3} M\,dx + N\,dy$$

How to find a potential function?

If $\frac{\partial f}{\partial x} = M$, $\frac{\partial f}{\partial y} = N$, do partial integration

E.g., if $\vec{F} = \langle y^2+1, 2xy \rangle$, have

$$\frac{\partial f}{\partial x} = y^2+1 \Rightarrow f = x(y^2+1) + C(y)$$

arbitrary fn depending only on y

$$\frac{\partial f}{\partial y} = 2xy \Rightarrow f = xy^2 + D(x)$$

$$\Rightarrow f(x,y) = xy^2 + x \quad (+\text{ constant})$$

get a better example? $\vec{F} = \langle ye^{xy} - 2x, xe^{xy} + 2y \rangle$

<u>Next</u>: 3 dimensions

Much of what we've done is still true:

- given a scaler function $f(x,y,z)$, the associated gradient vector field is $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y, f_z \rangle$
- Given a curve C parametrized by $\vec{R}(t) = \langle x(t), y(t), z(t) \rangle$, $a \le t \le b$ & a scaler fn $f(x,y,z)$, have a line integral

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\,\underbrace{|R'(t)|}_{\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}}\,dt$$

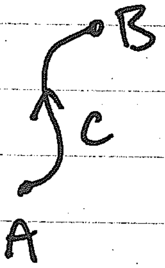
Likewise, have line integrals

$$\int_C M(x,y,z)\,dx + N(x,y,z)\,dy + P(x,y,z)\,dz$$

$$= \int_a^b \left(M\cdot\frac{dx}{dt} + N\cdot\frac{dy}{dt} + P\cdot\frac{dz}{dt} \right) dt.$$

The work done by a force $\vec{F} = \langle M,N,P \rangle$ moving a particle is still $\int_C \vec{F}\cdot d\vec{R}$ & the other expressions we wrote for this are still valid. (e.g., $W = \int_C \vec{F}\cdot\vec{T}\,ds$, etc)

- The FTLI is still valid:



$$\int_C \nabla f \cdot d\vec{R} = f(B) - f(A)$$

Thus also gradient fields are path-ind., & integrals around closed curves yield 0.

Likewise, it is still true that the converse holds.

Our "simple condition" for checking whether $\vec{F}$ is a gradient field gets more complicated: we now have $\vec{F} = \langle M(x,y,z), N(x,y,z), P(x,y,z)\rangle$ & must check $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, $\frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}$ and $\frac{\partial P}{\partial x} = \frac{\partial M}{\partial z}$; if any of these <u>fail</u> then <u>NOT</u> conservative, while if they all hold in a simply-connected region then $\vec{F}$ is conservative there.