could accidentally pick pts & curves on which the two integrals work out the same. // 2nd idea: guess f.

Another idea: if $\vec{F} = \nabla f$ for a sufficiently nice f, then
$\overset{\parallel}{\langle M, N\rangle}$

$$M = \frac{\partial f}{\partial x}, \quad N = \frac{\partial f}{\partial y} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

In our case, $\frac{\partial}{\partial y}(xy) = x \neq y^2 = \frac{\partial}{\partial x}(xy^2)$

So to test if a field is conservative, just compare appropriate partials

<u>E.g.</u> Is $\vec{F} = x^3y\,\hat{i} + xy^3\,\hat{j}$ conservative?

what about $\langle e^y\cos x, e^y\sin x\rangle$?

<u>If so</u>, what's the potential function f ?

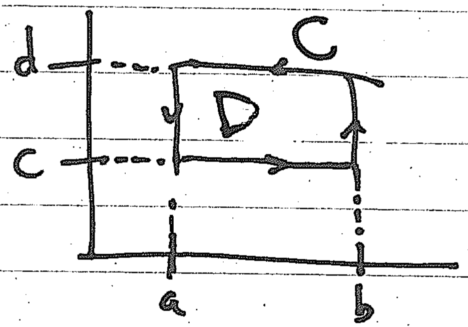
Question: know gradient field $\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Would be nice if the converse held, too.

Does it? We would need to know that

if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then integrals around closed loops are zero. In the process of studying this question, we will discover the 1st of 3 big thms of vector calculus: Green's Theorem.

Start w/ a rectangular closed curve:



& consider $\oint_C \vec{F} \cdot d\vec{R} =$

$\oint_C M\,dx + N\,dy =$

$$= \int_a^b M(x,c)\,dx + \int_c^d N(b,y)\,dy + \int_b^a M(x,d)\,dx + \int_d^c N(a,y)\,dy$$

$$= \int_c^d \big(N(b,y) - N(a,y)\big)\,dy - \int_a^b M(x,d) - M(x,c)\,dx$$

Observe $N(b,y) - N(a,y) = N(x,y)\Big]_{x=a}^{b} = \int_a^b \frac{\partial N}{\partial x}\,dx$

& sim. $M(x,d) - M(x,c) = \int_c^d \frac{\partial M}{\partial y}\,dy$

$$\Rightarrow \oint_C \vec{F}\cdot d\vec{R} = \int_c^d\int_a^b \frac{\partial N}{\partial x}\,dx\,dy - \int_a^b\int_c^d \frac{\partial M}{\partial y}\,dy\,dx$$

$$= \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$$

In particular, if $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ then this is 0.

Are we done? We've shown $\left(\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \Rightarrow\right.$ integrals over closed <u>rectangular</u> curves are $\left.0\right)$; what about other shapes?

<u>Thm</u> (Green): if C is piecewise smooth, simple closed curve bounding a region D, & if $M(x,y)$ & $N(x,y)$ are continuous w/ continuous partials along C & throughout D, then

$$\oint_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

Consequence: if a region R has no holes, & $\vec{F} = M\hat{\imath} + N\hat{\jmath}$ is defined throughout R, then

$\vec{F}$ is conservative on R if & only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Two things to do: examples, & prove Green's thm.

Ex 1: Conservative field: $\vec{F} = (3x^2+y^2)\hat{i} + 2xy\hat{j}$

Yes or no?

Ex 2: $\oint_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ around unit circle centered @ (0,0).

Ex 3: Evaluate $\oint_C (2y + \sqrt{1+x^4})dx + (5x - e^{y^2})dy$

where C is the circle $x^2+y^2=4$ by using

Green's Theorem to write it as a double integral.

Ex 4: Suppose R is a simply-connected region w/ boundary curve C.

Compute $\frac{1}{2}\oint_C -y\,dx + x\,dy$.

22.1

Proving Green's Theorem: In the case of a

vertically & horizontally simple region,