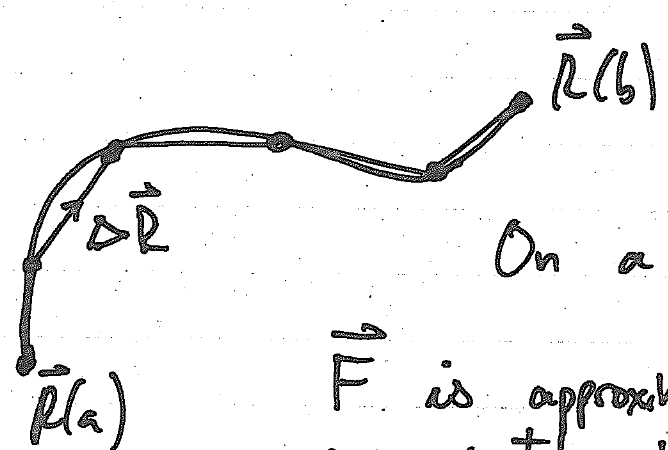
Recall: variable force field $\vec{F}(x,y) = M(x,y)\hat{i} + N(x,y)\hat{j}$

curve C given by parametric equations

$\vec{R}(t) = \langle x(t), y(t) \rangle$, $\quad a \le t \le b$.

Q: work done by the field moving a particle from $\vec{R}(a)$ to $\vec{R}(b)$?

Idea: break C into many small pieces:



On a little piece $\Delta\vec{R}$, $\vec{F}$ is approximately constant & C is approximately linear

$\Rightarrow \quad \Delta W \approx \vec{F} \cdot \Delta\vec{R}$

$\Rightarrow \quad dW = \vec{F} \cdot d\vec{R}$

$= \langle M, N \rangle \cdot \langle dx, dy \rangle$

$\Rightarrow \quad W = \int_C M\,dx + N\,dy = \int_C \vec{F} \cdot d\vec{R}$

We can also write as an arclength integral:

$$d\vec{R} = \frac{d\vec{R}}{ds} \cdot ds, \quad \& \quad \frac{d\vec{R}}{ds} = \hat{T}, \text{ the unit}$$

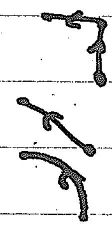
tangent, so $W = \int_C \vec{F} \cdot \hat{T}\, ds$.

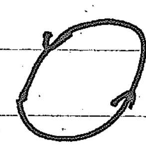
(The integral of the tangential component along C)

Can also rewrite in terms of t:

$$W = \int_a^b \left(M(x(t), y(t)) \frac{dx}{dt} + N(x(t), y(t)) \frac{dy}{dt} \right) dt.$$

E.g.: Suppose $\vec{F}(x,y) = y\hat{i} + (x+2y)\hat{j}$,

& $C = \Big\{$  Then $\int_C \vec{F} \cdot d\vec{R} = ?$

Notational point of interest: sometimes we'll integrate over a closed curve  ; then we may write $\oint_C \vec{F} \cdot d\vec{R}$ or $\oint_C M\,dx + N\,dy$

The

Let's look @ the preceding example from a different point of view: notice that

$$y\hat{i} + (x-2y)\hat{j} = \nabla f(x,y) \quad \text{where}$$

$$f(x,y) = xy - y^2. \quad \left(\text{check: } \frac{\partial f}{\partial x} = y, \ \frac{\partial f}{\partial y} = x-2y\right)$$

Thus
$$\int_C \nabla f \cdot d\vec{R} = \int_a^b \left(\frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}\right) dt$$

$$= \int_a^b \left(\frac{d}{dt} f(x(t),y(t))\right) dt$$

$$= f(x(b),y(b)) - f(x(a),y(a))$$

& <u>this answer doesn't depend on the path C except for its endpoints</u>

<u>Theorem</u>: (F.T.L.I.) If C is a curve going from A to B then $\int_C \nabla f \cdot d\vec{R} = f(B) - f(A)$. (where f is a diff'ble scalar field, C piecewise smooth)

<u>Corollary</u>: If C is a closed curve & $\vec{F}$ is a gradient field then $\oint_C \vec{F} \cdot d\vec{R} = 0$

& the converses are true as well:
~~(feedback and examples?)~~

if a field $\vec{F}$ is path-independent,

<u>define</u> $f(x,y) = \int_{(x_0,y_0)}^{(x,y)} \vec{F} \cdot d\vec{R}$

(check: really defines a fn)

Thus $f(x,y) = \int_{(x_0,y_0)}^{(x,y)} M\,dx + N\,dy$

boring; maybe skip?

$$\Rightarrow f(x+\Delta x, y) - f(x,y) = \int_{(x,y)}^{(x+\Delta x, y)} M\,dx$$

$$= \int_{x}^{x+\Delta x} M\,dx$$

$$\Rightarrow \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = \frac{1}{\Delta x}\int_{x}^{x+\Delta x} M\,dx$$

$\lim_{\Delta x \to 0}$

$$\downarrow \qquad\qquad \downarrow$$

$$\frac{\partial f}{\partial x} \qquad\qquad M$$

Obviously a lot of physics terminology going on here — let's add some more: $\vec{F}$ a force field & a particle of mass m is moved along curve C by this field from A to B.

$\hookrightarrow \langle x(t), y(t) \rangle,\ a \le t \le b$

$$W = \int_C \vec{F} \cdot d\vec{R} = \int_a^b \vec{F} \cdot \frac{d\vec{R}}{dt}\, dt$$

Newton: $\vec{F} = m\dfrac{d\vec{v}}{dt}$, $\left(\vec{v} = \dfrac{d\vec{R}}{dt}\right)$ so

write $\vec{F} \cdot \dfrac{d\vec{R}}{dt} = m\dfrac{d\vec{v}}{dt} \cdot \vec{v}$

$$= \frac{1}{2} m \frac{d}{dt}(\vec{v} \cdot \vec{v})$$

$$= \frac{m}{2} \frac{d}{dt}\left(|\vec{v}|^2\right)$$

$$\Rightarrow W = \frac{1}{2} m \int_a^b \frac{d}{dt}(|\vec{v}|^2)\, dt = \frac{1}{2} m |\vec{v}_B|^2 - \frac{1}{2} m |\vec{v}_A|^2$$

$$= \Delta(\text{Kinetic energy})$$

Now, suppose $\vec{F}$ is a gradient field,

$\vec{F} = \nabla f = -\nabla V$ (just take $V = -f$)

Then $\int_C \vec{F} \cdot d\vec{R} = -(V(B) - V(A)) = V(A) - V(B)$,

so $V(A) - V(B) = \frac{1}{2} m |\vec{v}_B|^2 - \frac{1}{2} m |\vec{v}_A|^2$

$$\Rightarrow \frac{1}{2} m |\vec{v}_A|^2 + V(A) = \frac{1}{2} m |\vec{v}_B|^2 + V(B)$$

So kinetic + potential energy is conserved under the action of this field

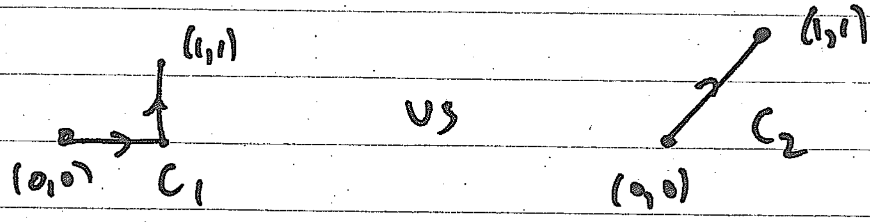
$\Rightarrow$ we call gradient fields conservative fields.

Know Gradient field $\Leftrightarrow$ conservative field $\Leftrightarrow$ path-independent $\Leftrightarrow$ integral round closed curves is 0

but how to recognize conservative fields?

E.g. Show $xy\,\hat{i} + xy^2\,\hat{j}$ not conservative

Pf: Show integral along 2 different curves joining same pts is different



However, this method is not fail-safe: you