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Menu: educ'n – OpenCourseWare – Science · Math – 18.01 (2006) – Readings – course reader (links)

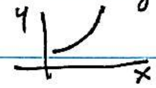
OCW website for supp. Notes (or can buy)

Lecture 1

1. Functions: represented by:

① formulas: $y = c_1 x$ $\quad y = 3x^2$, $\quad y = 5e^x \quad y = \sin 2x$

② geometrically:



③ table of data:

x	1	2	3	4
y	1.1	1.3	.9	-

Notations: $y = f(x)$ — x ind., y dep.

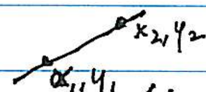
also: f, $f(x)$, $y(x)$

④ black box: $\xrightarrow{x} \boxed{f} \xrightarrow{f(x)}$

⑤ calculator buttons $\boxed{x}$ $\boxed{\sin}$

2. Derivatives:

slope of line:



$\dfrac{y_2 - y_1}{x_2 - x_1} = m$ constant

slope of curve? at (x_0, y_0)

$y = f(x)$, $y_0 = f(x_0)$

$:=$ slope of tang. line at (x_0, y_0) $= \lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$

$=$ lim of slope of secant $= \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \dfrac{dy}{dx}$

$y = x^n$ ($n \geq$ integer)

$y = \dfrac{1}{x}$

Example: $y = x^2$ at (x_0, y_0)

Notations: y', $\dfrac{dy}{dx}$, $f'(x)$, f', Df

(depending on which + whether variables are being used)

Derivative as Rate of change: w.r.t. time t

$y = \frac{1}{2} g t^2$ (falling body): $\dfrac{dy}{dt} = gt$ = velocity at time t

w.r.t. distance:

metal rod $\quad T(x)$ = temperature at x

$T'(x)$ = temperature gradient.

w.r.t. time $\quad A = A_0 e^{rt}$

growth law of bank acc't, t = years, r = interest rate

3. Differentiation rules

Two kinds: ① For specific functions: $Dx^k = k x^{k-1}$ any real k $\left[x^{-a} = \dfrac{1}{x^a}\right]$

(each has to be calc. from definition)

$D \sin x = \cos x$

$\cos x = -\sin x$

$D e^x = e^x$

② Sum, product, quotient, chain rules

$(uv)' = u'v + v'u$

Ex: $(x^2+4)^{-1}$

$\left(\dfrac{u}{v}\right)' = \dfrac{vu' - uv'}{v^2}$

Ex: $\dfrac{1}{x^3}$ $\quad \dfrac{x^2-1}{x^2+1}$

$\sqrt{x^2+1}$

$(x^2+1)^{100}$

$u(v(t))' = u'(v(t)) \cdot v'(t)$ — composite function

$y = u(x)$, $x = v(t)$ $\quad \dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt}$

Applications: higher derivatives

① Rates of change: $y = f(t)$ position

$\frac{dy}{dt} = y' = f'(t)$ velocity

$\frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2} = y'' = f''(t)$ accel., etc.

② Studying graphs: by calculating derivatives

$f' > 0$ | $f'' > 0$ (gives 2^{nd} deriv test for max-min)

$f' < 0$ | $f'' < 0$

$f' = 0$ | $f'' = 0$

1^{st} deriv.

Ex: $y = x^3 - 3x + 2$

③ Finding tan. lines: $y = \sqrt{25 - x^2}$ at $(3,4)$

Directly: Find y' at $(3,4)$

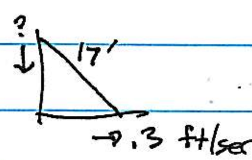
Implicitly: $x^2 + y^2 = 25$ use implicit diff'n

~~$-y^3 - y + x^2 + x = 0$ tan line at (2,2)?~~

Modeling problems: { Draw picture; Choose variables [often not x, y !!!]; Find relations between them } (and use derivatives)

① Max-min: open box, square base, vol = 250, with least surface area? (check using 2^{nd} deriv)

② Related rate: ladder

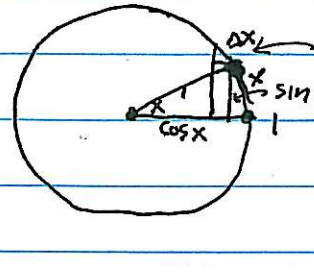


~~③ Probs leading to simple diff. eqns (very important: needs more technique)~~

Started trig functions, but repeated in lecture 2

mainly: $D \sin x = \cos x$

by geometric reasoning:



similar triangles

$\Delta(\sin x)$

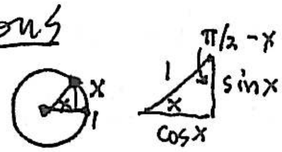
$\frac{d \sin x}{dx} \approx \frac{\Delta(\sin x)}{\Delta x} \approx \cos x$

Lecture 2

Know standard values! use → 

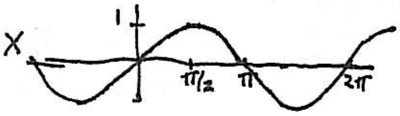
Trig functions

$\sin x$:
$\cos x$



x = arclength
$\sin^2 x + \cos^2 x = 1$

$\sin x$



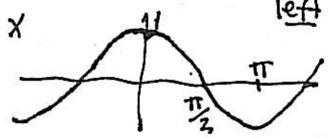
$\sin(-x) = -\sin x$ odd function

$D \sin x = \cos x$ (by similar triangles)

$D \cos x = -\sin x$ (by ⊛ + chain rule)

⊛ $\cos x = \sin(x + \pi/2)$ ↑ moves graph to left (see Notes G)

$\cos x$



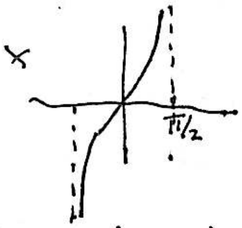
period 2π

Others: $\tan x = \dfrac{\sin x}{\cos x}$ $\sec x = \dfrac{1}{\cos x}$

$D \tan x = \sec^2 x$ (quotient rule)

$D \sec x = \sec x \tan x = \dfrac{\sin x}{\cos^2 x}$

$\tan x$



period π

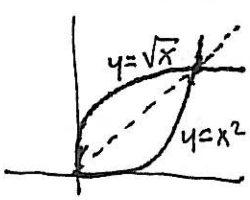
Note: Radians used to avoid constants in differentiation

let $\mathcal{S}\text{in}\, x = \sin\left(\frac{\pi}{180} x\right)$ (x ↑ degrees, ↑ radians)
[check for $x = 90, 180, 360$]

$\frac{d}{dx} \mathcal{S}\text{in}\, x = \frac{\pi}{180} \cos\left(\frac{\pi}{180}x\right)$
$= \frac{\pi}{180} \mathcal{C}\text{os}\, x$

Inverse functions

Example



$y = x^2$
$x = y^2$ flip about $y = x$
same as
$\sqrt{x} = y$ inverse function to x^2

(domain must be restricted to $x \geq 0$)

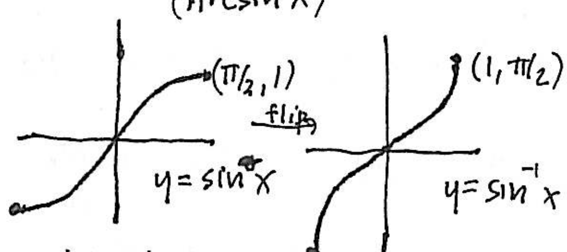
Differentiation: (use implicit diff'n)

$y = \sqrt{x} \xrightarrow[\text{same}]{} y^2 = x$
$\downarrow D$
$2yy' = 1$
$y' = \frac{1}{2y}$
$y' = \frac{1}{2\sqrt{x}} \longleftarrow$

same scheme works for all inverse functions

Inverse trig functions (only two are important)

$y = \sin^{-1} x$
(Arcsin x)

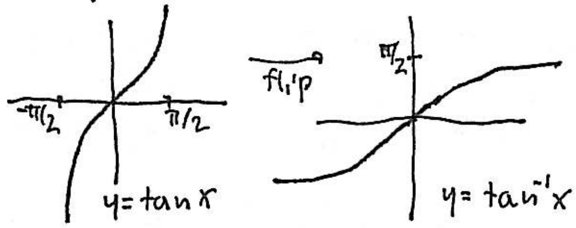


restrict domain to $[-\pi/2, \pi/2]$

$x \underset{\sin}{\overset{\sin^{-1}}{\rightleftarrows}} y$

Diff'n: $y = \sin^{-1} x \xrightarrow[\text{same}]{} \sin y = x$
$\downarrow D$
$\cos y \cdot y' = 1$
$y' = \frac{1}{\cos y}$
$y' = \frac{1}{\sqrt{1 - x^2}} \longleftarrow$

$y = \tan^{-1} x$ (Arctan x)



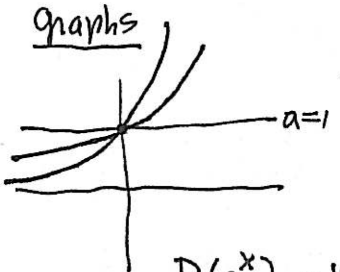
$y = \tan^{-1} x \longrightarrow \tan y = x$
$\downarrow D$
$\sec^2 y \cdot y' = 1$
$y' = \frac{1}{\sec^2 y}$
$y' = \frac{1}{1 + x^2} \longleftarrow$

Exponential Function a^x

$a > 1$ $a =$ "base"

(2, e, 10 are usual choices)
($2 \uparrow$ CS, $e \uparrow$ calc, $10 \uparrow$ non-calc. sci + eng)

graphs



Exp. laws: $a^0 = 1$

$a^{x_1+x_2} = a^{x_1} a^{x_2}$ "exp'l law"

$(a^{x_1})^{x_2} = a^{x_1 x_2}$

$$D(a^x) = \lim_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x} = a^x \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$$

$f(x) = a^x$

$$= a^x \cdot f'(0)$$

e is the base for which $f'(0) = 1$ (slope of a^x at $x=0$ is 1)

$\therefore D(e^x) = e^x$

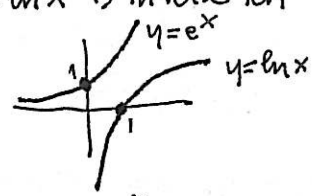
Inverse function

$x \xrightarrow{e^x} y$, $\xleftarrow{\ln}$

$y = e^x \iff \ln y = x$ $y = \ln x$ is inverse fcn

$\therefore$ $\boxed{\begin{array}{l} y = e^{\ln y} \\ \ln(e^x) = x \end{array}}$ BASIC inverse relations

$\therefore \ln e = 1$



Derivative: $y = \ln x \longrightarrow e^y = x$
$\downarrow D$

GIVEN AS SEATWORK

$e^y \cdot y' = 1$

$y' = \frac{1}{e^y}$

$y' = \frac{1}{x} \longleftarrow$

SEATWORK PROBLEM

Derivative of a^x:

$$D a^x = D(e^{\ln a})^x$$
$$= D e^{x \ln a}$$
$$= e^{x \ln a} \cdot \ln a$$
$$D a^x = a^x \ln a$$

DIDN'T DO

Alternate derivation

$y = a^x$

$\ln y = x \ln a$

$\frac{1}{y} \cdot y' = \ln a$

$y' = a^x \ln a$

log law: $y_i = a^{x_i}$, $x_i = \ln y_i$

$a^{x_1+x_2} = a^{x_1} \cdot a^{x_2}$

$= y_1 y_2$

$\therefore x_1 + x_2 = \ln(y_1 y_2)$

$\ln y_1 + \ln y_2 = \ln(y_1 y_2)$

Similarly:

$\ln a^x = x \ln a$

DIDN'T DO [Useful numbers: $\ln 10 \approx 2.3$
$e \approx 2.718$ $\ln 2 \approx .7$
$\ln e = 1$

- In differentiating: always make the base e. (or use logarithmic diff'n ("alternative") above)

DIDN'T DO — GAVE FOR HW.

In engineering: $\log x$ usually means $\log_{10} x$

$a^{\log_a x} = x$

Take ln: $\log_a x \cdot \ln a = \ln x$

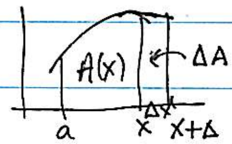
$\log_a x = \frac{\ln x}{\ln a} \approx \frac{\ln x}{2.3}$ (use $a = 10$)

~~Velocity integr~~ ~~x = time~~

A_a^b : problem: find area under $f(x) > 0$, over $[a, b]$

idea: $A(x)$ — make it a function instead of a number — functions are easier to find since they can be differentiated

$$\Delta A \approx f(x)\,\Delta x$$



$$\therefore \frac{\Delta A}{\Delta x} \approx f(x)$$

$\Delta x \to 0$ $\downarrow$

Basic procedure $\qquad \dfrac{dA}{dx} = f(x)$ $\quad$ ODE

IVP add: $A(a) = 0$ $\quad$ to get a unique answer

Find areas $\qquad$ Find an $F(x)$ s.t. $\quad$ [called an "anti-deriv" of $f(x)$ or an indefinite integral]

$$\frac{dF}{dx} = f(x)$$

Theorem:

if $f'(x) = 0$ on $[a,b]$, then $f(x) = c$ on $[a,b]$. (non-trivial to prove)

then $\dfrac{d(A-F)}{dx} = 0 \quad \therefore A(x) = F(x) + c \quad$ find c?

$$A(a) = F(a) + c$$

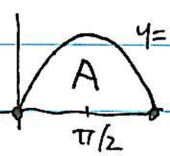
$$c = -F(a)$$

Area: $\displaystyle\int_a^b f(x)\,dx$ $\qquad \therefore A(x) = F(x) - F(a)$

$$\boxed{A(b) = F(b) - F(a)}$$

$\int$ notation: $\qquad = F(x)\Big]_a^b = \int f(x)\,dx\Big]_a^b$

Practice
Finding areas:

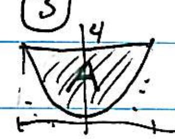


① $A = \displaystyle\int_0^{\pi} \sin x\,dx = -\cos x\Big]_0^{\pi} = -\left[\cos\pi - \cos 0\right] = -\left[-1-1\right] = 2$

seatwork problems:

~~$\int_0^{2\pi} = 0$ negative area = below x axis.~~ not done

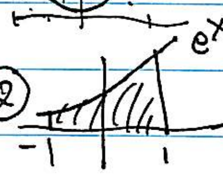
③



~~$\int_0^{\pi/2} \sin 2x\,dx = -\frac{1}{2}\cos 2x\Big]_0^{\pi/2}$~~ $=$

$u = 2x$

②



④ under $y = \sqrt{1-x^2}$ over $[-1, 1]$

Discussion:

$\int_{-1}^{1} \sqrt{1-x^2}\,dx = \frac{\pi}{2}$ (area of unit semicircle)
— harder to do by fund. thm.

$\int_0^1 x\sqrt{1-x^2}\,dx$: this is actually easier.

Finding antiderivatives (basic ones):

Will learn more sophisticated method.

For starting ones, basic method is:

① Guess the answer, differentiate it, and correct the constant

② Use direct substitution: $u = (\overset{x-}{\text{blob}})$

$du = (x\text{-blob})' \cdot dx$

Ex: $\int \cos(2x)\,dx$ Guess: $\sin 2x \xrightarrow[D]{} 2\cos 2x$ $\therefore$ use $\frac{1}{2}\sin 2x$

or $\left.\begin{aligned} u &= 2x \\ du &= 2\,dx \end{aligned}\right\} = \int \cos u \frac{du}{2} = \frac{1}{2}\sin u$

$= \frac{1}{2}\sin(2x)$

Ex: $\int x\sqrt{1-x^2}\,dx$ Guess: $(1-x^2)^{3/2}$, adjust constant by differentiating

or: $\left\{\begin{aligned} u &= 1-x^2 \rightsquigarrow du = -2x\,dx \\ u &= x^2 \end{aligned}\right.$

plug in, ~~$u = (1-x^2)^{3/2}$~~

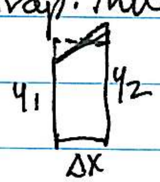
get $u^{3/2}$, etc. change back to x at end

→ INSERT NEXT PAGE

· DID THIS LAST

↳ Numerical calculation if no antiderivative:

Trap. rule



area $= \frac{y_1+y_2}{2} \cdot \Delta x$

average ht · width

leads to $\int_a^b f(x)\,dx = \Delta x\left(\frac{y_0}{2} + y_1 + \dots + y_{n-1} + \frac{y_n}{2}\right)$

Definite integral $\int_a^b f(x)\,dx$

This has to be defined as ~~a~~ limit of a sum, since areas are too restrictive — it is evaluated by the fund. theorem, or numerically, but when ~~used~~ set up to calculate something, one has almost always to go back to the sum definition

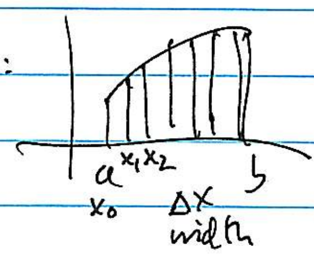
General procedure:

① Divide what you want to calculate into n little pieces that you can calculate approximately

② Add them up: to get a total ~~$\sum f(x_i)$~~

③ let $n \to \infty$

Applied to areas:



little pieces: $f(x_i)\Delta x$

Add up: $\sum_1^n f(x_i)\,\Delta x$

$\Delta x \to 0$
$n \to \infty$ $\downarrow$
$\int_a^b f(x)\,dx$

procedure will be used to calculate much more complicated things than simple areas under curves (both in review 18.01, and in 18.02)

Numerical integration (Trap. rule)

lecture 3 ?

① Simple volumes of rotation; discs, shells volumes by
to illustrate division into little pieces

Inverse Substn, Integ by parts Partial fracs. ② Mixture of 3 techniques of integration
and applications like work, average value
and stuff like in
Exercises 4J

[non-physics

[For partial fractions look at Notes H — they don't need any more than that (I often don't even do completing the square)

lecture 4 ?

improper integrals
∞ series, approx, L'Hosp's