Line integrals

Suppose we have a curve C given by parametric eqns $x = x(t)$, $y = y(t)$, $a \le t \le b$. Given a function f, we may integrate f over C via

$$\int_C f(x,y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

("area of a curtain") ("summing the values of f on the curve C")

Note that this value is parametrization-independent — only the curve C & function f & direction on C matter. It is the line integral w.r.t. arclength.

We could also take different kinds of line integrals, like line integrals w.r.t. x. What would this mean?

$$\int_C f(x,y)\, dx = \int_a^b f(x(t), y(t))\, x'(t)\, dt$$

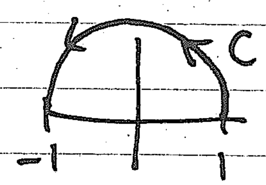
& similarly can consider $\int_C f(x,y)\,dy = \int_C \left(f(x(t),y(t))\frac{dy}{dt}\right) dt.$

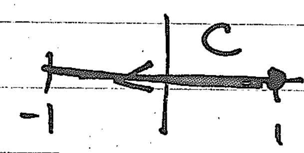
Often these appear together, as in

$$\int_C P(x,y)\,dx + Q(x,y)\,dy := \int_C \Big(P(x(t),y(t))x'(t) + Q(x(t),y(t))y'(t)\Big)\,dt$$

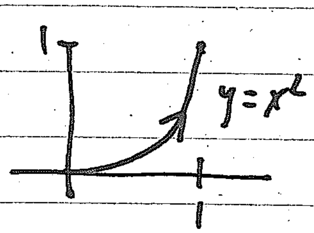
These definitions may not seem natural, but we'll see why we need them shortly.

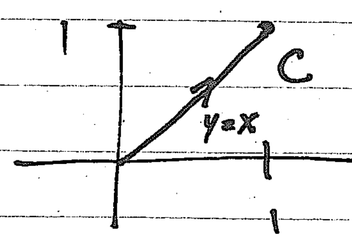
E.g. Compute $\int_C (2+x^2y)\,ds$ where

a) 

b) 

E.g Compute $\int_C x^2y\,dx + (x-2y)\,dy$ where

a) $C =$ 

b) 

Vector fields

A vector field assigns to every point in $\begin{cases} \text{the plane} \\ \text{space} \\ \text{a 2D-region} \\ \text{a 3D region} \end{cases}$ a vector (rather than a number) — just a vector function.

Examples: • force fields in physics (like gravitational field)
• fluid flows (velocity vector fields)

(As usual, assumptions that everything is continuous, diff'ble, etc.

Write $\vec{F}(x,y) = M(x,y)\,\hat{i} + N(x,y)\,\hat{j}$

$$\vec{F}(x,y,z) = \left\langle M(x,y,z),\ N(x,y,z),\ P(x,y,z) \right\rangle$$

Graphing?

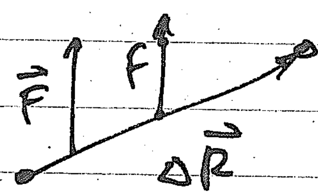
For the moment, we'll focus on the 2-D case.

Have seen one "math" example of a vector field: given a scalar multivariate function $f(x,y)$, the gradient $\nabla f(x,y) = \left\langle f_x(x,y),\ f_y(x,y) \right\rangle$ is a vector field.

(One way to think of such a situation: f is a <u>potential energy</u> function (or abbrev. <u>potential function</u>) & ∇f is the associated force field.)

Physical problem: how much work is done by a force moving an object?

Well, for a constant force & linear motion of the object,



it's just work $= \vec{F} \cdot \Delta \vec{R}$

What about more complicated situations, where $\vec{F}$ may not be constant & the curve of motion
$\vec{F}(x,y)$
may not be linear?