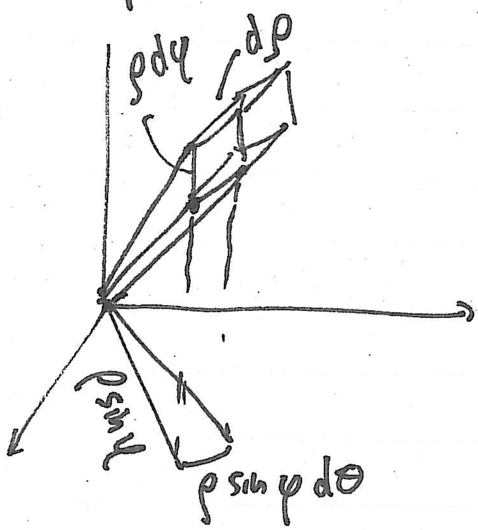
Ex 2: triple integral to find the volume of a sphere

$$2\iint_{x^2+y^2\le a^2}\int_0^{\sqrt{a^2-x^2-y^2}} dz\,dx\,dy = \iint_{\substack{0\le r\le a\\ 0\le\theta\le 2\pi}}\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{a} r\sqrt{a^2-r^2}\,dr\,d\theta = \cdots$$

Spherical coordinates & triple integrals



$$dV = (\rho\,d\varphi)\cdot(d\rho)\cdot(\rho\sin\varphi\,d\theta)$$

$$= \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$\Rightarrow \iiint_R f(x,y,z)\,dV =$$

$$= \iiint_R f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

In any particular problem, the usual hard part is to find the boundary of the region to express as an iterated integral.

Ex

A hollow sphere has outer radius 3, inner radius 2, & density proportional to distance from the origin.

Mass = ?

Finally, a quick remark about changing coordinates:

Suppose we have a region R of the plane & we reparametrize by $x = x(u,v)$, $y = y(u,v)$.

How do we write $\iint_R f(x,y)\,dx\,dy$ in terms of u, v, du, dv?

(Recall in 1-var case have $\displaystyle\int_a^b f(x)\,dx = \int_c^d f(x(u))\,\frac{dx}{du}\,du$)

where $a = x(c)$, $b = x(d)$
(when $u = c$, $x = a$)

Theorem: Suppose S is a region in the u,v-plane & that $(x(u,v), y(u,v))$ is a 1-to-1, continuous, differentiable function to the x,y-plane under which the image of S is R. Then

$$\iint_R f(x,y)\, \overset{dx\,dy}{\overset{\downarrow}{dA}} = \iint_S f(x(u,v), y(u,v)) \cdot \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right| du\, dv$$

absolute value

Jacobian of the transformation $u,v \mapsto x,y$. $\frac{\partial(x,y)}{\partial(u,v)}$

Example: polar to rectangular

$$x(r,\theta) = r\cos\theta$$
$$y(r,\theta) = r\sin\theta \quad \Rightarrow \quad \frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$
$$= r \quad \checkmark\checkmark$$

Same deal for 3D: define

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \cdots & \frac{\partial x}{\partial w} \\ \vdots & \ddots & \vdots \\ \frac{\partial z}{\partial u} & \cdots & \frac{\partial z}{\partial w} \end{bmatrix}$$

Then

$$\iiint_R f(x,y,z)\, dV = \iiint_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw$$

E.g. Spherical/rectangular: