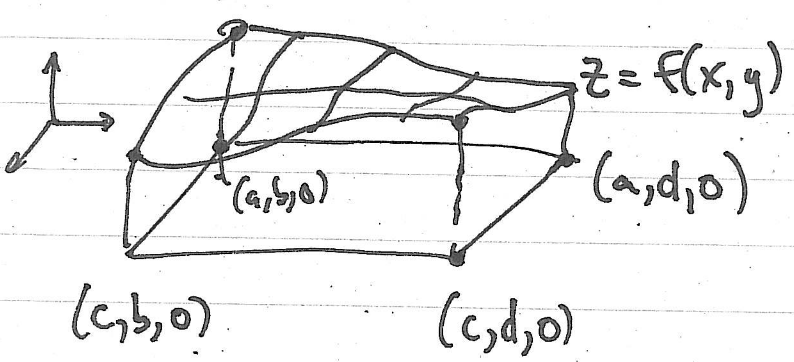
Final note on max-min problems

Absolute max & min over some closed region: first look for crit pts on interior; then look for "constrained extrema" on boundary.

Integation of multivariate functions

Double integrals & iterated integrals

Surface S lies over some rectangle R. Volume below S & above R = ?



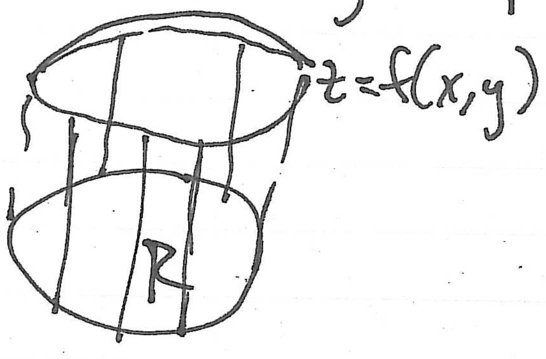
Idea: cut into slices.

$$dV = A_{slice}\, dy = \left(\int_a^c f(x,y)\, dx \right) dy$$

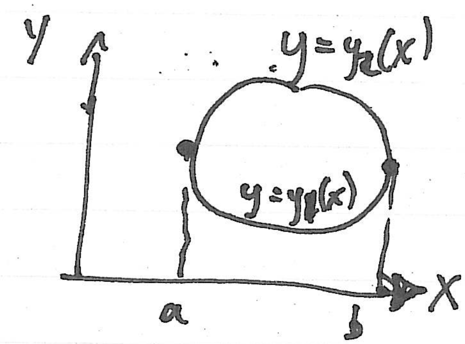
area of the slice w/ fixed y-coordinate

$$\Rightarrow V = \int_b^d \int_a^c f(x,y)\, dx\, dy$$

Note that the fact that R was a rectangle wasn't really important



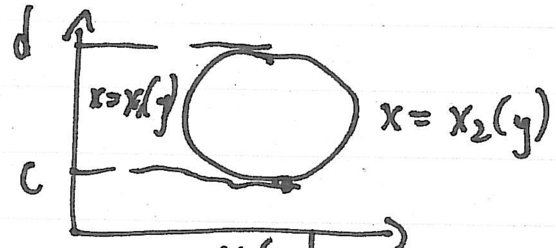
$R=$



$$A(x)=\int_{y_1(x)}^{y_2(x)} f(x,y)\,dy$$

$$V=\int_a^b A(x)\,dx = \int_a^b \int_{y_1(x)}^{y_2(x)} f(x,y)\,dy\,dx$$

<u>OR</u>



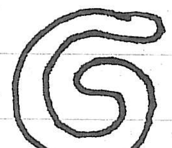
$$A(y)=\int_{x_1(y)}^{x_2(y)} f(x,y)\,dx \qquad V=\int_c^d \int_{x_1(y)}^{x_2(y)} f(x,y)\,dx\,dy$$

This is called using <u>iterated integrals</u> to find volume.

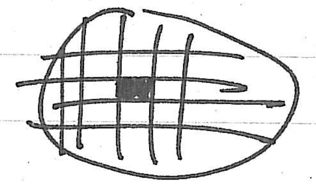
<u>E.g.</u> The iterated integral $\int_0^1 \int_{x^2}^{x} 2y\,dy\,dx$

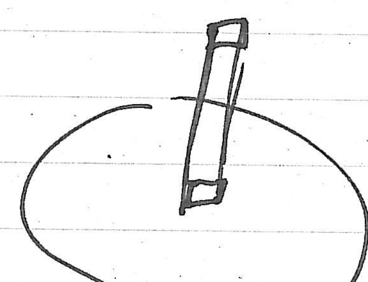
is a volume above some region of the plane. Find this region, reverse order of integration, & compute the volume in both orders.

In general, not every integral over a region can be written as a single iterated integral, e.g.

 = R. More general approach is via

<u>double integrals</u>: cut region up into many little ~~slices~~ pieces (not nec. slices), then add up





Write this $\iint_R f(x,y)\,dA$. (little pieces of area)

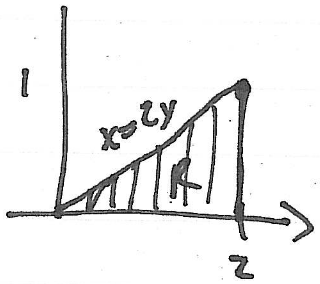
When region is "vertically simple" can write this as an iterated integral $dy\,dx$; & "horiz. simple;" as an iterated integral $dx\,dy$.

$dx \, \overset{dy}{\boxed{dA}}$

(in both cases, we're using $dA = dx\,dy = dy\,dx$ – in other coordinate systems, need to translate carefully between dA & $d(\text{coordinate vars})$.)

Maybe hold the second half of this comment until later

Ex: $R =$



$$\iint_R 4e^{x^2}\,dA$$

Some applications: given a flat disk of variable density $\delta(x,y)$, total ~~volume~~ mass is

$$M = \iint_R \delta(x,y)\,dA$$

Center of mass: $(\bar{x}, \bar{y})$ where

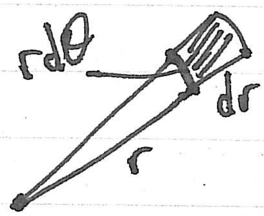
$$\bar{x} = \frac{\iint_R x \cdot \delta(x,y)\,dA}{M}, \quad \bar{y} = \frac{\iint_R y \cdot \delta(x,y)\,dA}{M}$$

Double integrals in polar coordinates

Sometimes regions are easier to describe in polar coords → how can we double integrate in polar?

$$\iint_R f(x,y)\,dA = \iint_R f(r\cos\theta, r\sin\theta)\,dA, \text{ but how to}$$

write dA in terms of dr, $d\theta$



$$dA = r\,dr\,d\theta = r\,d\theta\,dr$$

$\uparrow$ more usual order.

<u>Ex</u>: Compute the volume under the paraboloid $z = x^2+y^2$ & above the disk $x^2+y^2 \le 9$ using an ~~double~~ iterated integral in polar coordinates.

Parametric surfaces & surface area

So far, equations of surfaces have been given either in the form $z = f(x,y)$ or implicitly as $F(x,y,z) = 0$. Now, we consider a third way to describe a surface: parametric equations. Surface is 2D $\Rightarrow$ 2 parameters u, v. In vector form,

$$\vec{r}(u,v) = x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k}$$
$$= \langle x(u,v),\ y(u,v),\ z(u,v) \rangle$$

or alternatively by parametric eqns

$x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$.

where (u,v) varies through some region D (the domain) in the u,v-plane. (as w/ parametric curves, each choice of parameters gives a point of the surface; set of all choices $\rightsquigarrow$ whole surface)

E.g. $\vec{r}(u,v) = \langle 2\cos u,\ v,\ 2\sin u \rangle$

What's the surface?

16.1

E.g. $\vec{r}(u,v) = \langle x_0, y_0, z_0 \rangle + u\vec{a} + v\vec{b}$ where $\vec{a}, \vec{b}$ fixed 3D vectors (not parallel)

E.g. How to parametrize a torus?