E.g., $z = \frac{x}{x^2+y^2}$, $x = t\cos u$, $y = t\sin u$

$$\frac{\partial z}{\partial t} = ?$$

Ans: "What you would expect"

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

(& in general?)   (Can write this as

$\frac{\partial z}{\partial t} = \nabla z \cdot \frac{\partial}{\partial t}(x, y)$, but people don't seem to for some reason.)

14.1

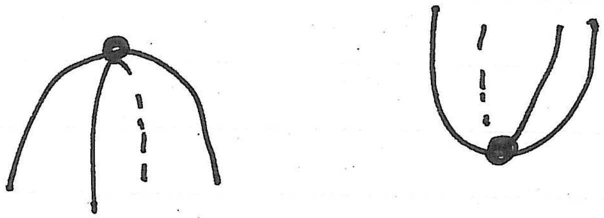
Max-min problems: locally optimizing a function of several variables

Two ways of looking at the situation:

- For (x_0, y_0) to be a max, $f(x, y_0)$ needs a max @ $x = x_0$ & $f(x_0, y)$ needs a max @ $y = y_0$, so $f_x(x_0, y_0) = 0$ & $f_y(x_0, y_0) = 0$.

- For (x_0, y_0) to be a max, need horizontal tangent plane $\Rightarrow$ $z = \text{const}$ $\Rightarrow$ $\nabla f = \vec{0}$.

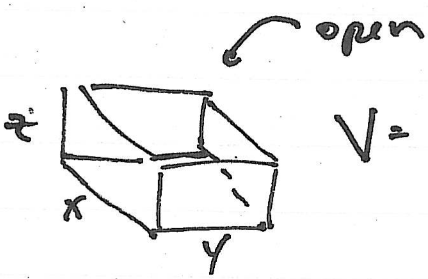
Same analysis for a minimum



$\boxed{\text{Critical points}}$

But could be max in x-direction, min in y-direction

⟵ saddle pt : $f_x = f_y = 0$
but not max nor min.

E.g.: 

$V = 4 \text{ ft}^3$ Min(surface area) = ?

Easy to tell is min. What about in more complicated situations like ~~$z = (y-x^2)(y-2x^2)$~~ ~~whatever~~.

$$z = 3xy^2 + y^2 - 3x - 6y + 7 \quad ?$$

<u>2^{nd} derivative test</u> : Analogue of 1^{st} derivative is <u>gradient</u> : critical points are $\nabla f = \vec{0}$.

14.3

Analogue of 2^{nd} derivative is the

<u>Hessian matrix</u> $\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$ and/or its

determinant $D = f_{xx} \cdot f_{yy} - (f_{xy})^2$.

2^{nd} derivative test states:

If (x_0, y_0) is a crit. pt. then

(1) $D(x_0,y_0) > 0$, $f_{xx}(x_0,y_0) < 0 \Rightarrow$ max

(2) $D(x_0,y_0) > 0$, $f_{xx}(x_0,y_0) > 0 \Rightarrow$ min

(3) $D(x_0,y_0) < 0 \Rightarrow$ saddle point

(4) $D(x_0,y_0) = 0 \rightarrow$????? (Need another tool.)

E.g. find & classify critical points of

$$z = 3xy^2 + y^2 - 3x - 6y + 7.$$

beyond the scope of our class

(In general: positive-definite $\Rightarrow$ local min
negative-definite $\Rightarrow$ local max
both positive & negative eigenvalues $\Rightarrow$ saddle pt
o/w $\Rightarrow$ ~~indeterminate~~ inconclusive)

Question: $f(x,y) = x^4 + y^4$ @ $(0,0)$?

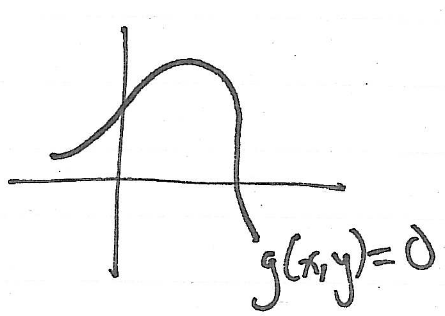
That was <u>unconstrained</u> max-min: over the entire domain, where does max or min occur? But, if we aren't allowed to go everywhere, the max & min values we can reach may change.

<u>E.g.</u>: Minimize x^4+y^4 subject to $x^2+y^2=1$.

How to do? (Do these methods work if it were $x^4+y^4+z^4$ on $x^2+y^2+z^2=1$?)
(Or what if the constraint function was too complicated to solve for z?)

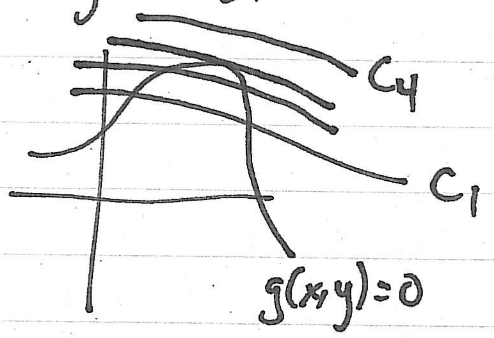
Let's tackle this problem in general:

Suppose we have a function $f(x,y)$ to maximize subject to the constraint $g(x,y)=0$.

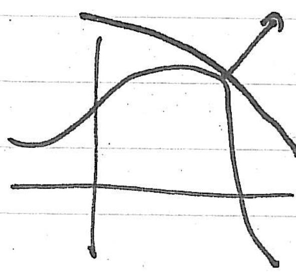


Want to find largest value of c s.t. $f(x,y)=c$ is achieved on this curve.

Draw some level curves $f(x,y) = c_1$, $f(x,y) = c_2$, etc. say $c_1 < c_2 < c_3 < c_4$



Max is achieved at a point of tangency — o/w, can "slide" to a slightly larger value. At a point of tangency, the normals are in the same direction, so the gradients of f & g



are parallel. That is, there is some λ s.t. $\nabla f = \lambda \cdot \nabla g$

So we must solve simultaneously the three equations $\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$, $\frac{\partial f}{\partial y} = \lambda \cdot \frac{\partial g}{\partial y}$ and $g(x,y) = 0$ for the variables λ, x, y (but actually we don't care about the value of λ).

E.g. Maximize x^4+y^4 subject to $x^2+y^2=1$.

$f(x,y) = x^4+y^4$ $\qquad$ $g(x,y) = x^2+y^2-1$

Solve $\quad 4x^3 = 2x\lambda, \quad 4y^3 = 2y\lambda, \quad x^2+y^2=1$

$(x=0$ or $\lambda = 2x^2)$ $\qquad$ $(y=0$ or $\lambda = 2y^2)$

$\downarrow$ $\qquad\qquad\qquad\qquad$ $\downarrow$

$(0,\pm 1)$ $\qquad\qquad\qquad$ $(\pm 1, 0)$

$x^2=y^2, \quad x^2+y^2=1 \quad \rightsquigarrow \quad \left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$ (4 points)

$\Rightarrow$ Max value 1 @ $(\pm 1, 0), (0,\pm 1)$
Min value $\frac{1}{2}$ @ $\left(\pm\frac{1}{2}, \pm\frac{1}{2}\right)$

Fact: same argument works for fns of more variables – geometrically, involve level surfaces instead of level curves.

E.g: Find point on plane $2x+3y+6z=1$ that minimizes the quantity $x^2+y^2+z^2$

(equiv: that minimizes distance to $(0,0,0)$)

Note: also exist versions for multiple constraints (check wikipedia)