We denote $\langle f_x, f_y \rangle = \text{grad } f = \nabla f$ $\boxed{12.3}$

$$= \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

(*) & so $D_{\hat{u}} f = \nabla f \cdot \hat{u}$ (& sim for 3-var or n-var)

Since $\hat{u}$ is a unit vector, this says "the directional derivative of f in direction $\hat{u}$ is the magnitude of the projection of ∇f onto $\hat{u}$."

So ∇f summarizes all the info. on direct. derivatives that we could want to know.

Have seen one fund. prop. of ∇f. Two more: $\boxed{13.1}$

$D_{\hat{u}} f = |\nabla f| \cos\theta \leq |\nabla f|$, with equality when

(*) $\theta = 0$, i.e., when $\hat{u}$ is in the same direction as ∇f

$\Rightarrow$ the vector ∇f points in the direction in

(*) which f increases most rapidly, & this max rate of increase is precisely $|\nabla f|$.

E.g. If $f(x,y,z) = x^2 - y + z^2$

dir. deriv @ $(1,2,1)$ in dir. $\langle 4,-2,4\rangle$

Max change? etc.

Finally, a 4th important property of the gradient.

Suppose we have a surface $z = f(x,y)$. Recall that it has level curves defined by setting $z = c$ for some constant c $\rightsquigarrow$ $f(x,y) = 3$, $f(x,y) = 0$, $f(x,y) = c$, etc. If point (x,y) moves along this curve, (say starting from some point (x_0,y_0)), then the value of f doesn't change (by definition).

$\Rightarrow$ the directional derivative of f in the direction tangent to the curve is $\underline{\underline{0}}$.

$\Rightarrow$ $\cos\theta = 0$, i.e., the gradient of f is $\perp$ to the curve $f(x,y) = \text{const.}$

An identical argument (using level surfaces $f(x,y,z)=c$ of a function of 3 variables) holds true in 3D case $\Longrightarrow$ ∇f is orthogonal to the surface defined by the implicit equation

$f(x,y,z) = c$ for any constant c.

Ex: Compute ∇f & $D_{\hat{u}} f$ for

- $f(x,y,z) = \ln(x^2 + y^2 + z^2)$, $\hat{u} = \frac{\langle 1,1,1 \rangle}{\sqrt{3}}$, @ $P = (1,3,-2)$
- $f(x,y,z) = e^{xyz}$ @ $P = (2,1,1)$ in the max'l direction.

Ex: find the tangent plane to the surface $xyz = 4$ at the point $(2,-2,-1)$ & the normal line there.

Chain rules for partial derivatives

Have already seen that

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy .$$ If have $z(x,y)$

Where in turn x, y are functions of t, can divide by dt to get

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} .$$

z is determined by t

E.g: If $z(x,y) = e^{x^2+y^2}$, $x = \cos t$, $y = \sin t$, what is $\frac{dz}{dt}$?

Similarly, for 3-variable fns, $w(x,y,z)$, if

$x = x(t)$, $y = y(t)$, $z = z(t)$ then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$

Another possible situation: (e.g., <u>change of coordinates</u>)

$z = z(x,y)$, $x = x(s,t)$, $y = y(s,t)$, how does z change as s changes?

E.g., $z = \dfrac{x}{x^2+y^2}$, $x = t\cos u$, $y = t\sin u$

$$\frac{\partial z}{\partial t} = ?$$

Ans: "What you would expect"

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

(& in general?) (Can write this as

$$\frac{\partial z}{\partial t} = \nabla z \cdot \frac{\partial}{\partial t}(x, y),$$ but people don't seem to for some reason.)

14.1

Max-min problems: locally optimizing a function of several variables

Two ways of looking at the situation:

- For (x_0, y_0) to be a max, $f(x, y_0)$ needs a max @ $x = x_0$ & $f(x_0, y)$ needs a max @ $y = y_0$, so $f_x(x_0, y_0) = 0$ & $f_y(x_0, y_0) = 0$.
- For (x_0, y_0) to be a max, need horizontal tangent plane $\Rightarrow$ $z = \text{const}$ $\Rightarrow$ $\nabla f = \vec{0}$.