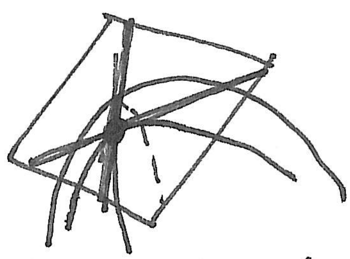
Tangent planes: If a surface is "smooth" (say, if it is given by $z=f(x,y)$ where f is continuous & has continuous partial derivatives) then it has tangent planes that "best approximate" the graph of the fn at each point. This plane must contain the two tangent lines mentioned earlier (those that arise when we cut with the planes $x=x_0$ & $y=y_0$)



Now, these lines had slopes (in the 2D-cuts) of $f_x(x_0,y_0)$ & $f_y(x_0,y_0)$, so they have direction vectors

$$\vec{V}_1=\langle 0,1,f_y(x_0,y_0)\rangle \quad \& \quad \vec{V}_2=\langle 1,0,f_x(x_0,y_0)\rangle$$

& the normal to the plane is given by

$$\vec{N}=\vec{V}_1\times\vec{V}_2=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & f_y(x_0,y_0) \\ 1 & 0 & f_x(x_0,y_0) \end{vmatrix}=\langle f_x(x_0,y_0),f_y(x_0,y_0),-1\rangle$$

Let $z_0=f(x_0,y_0)$. Then the plane has eqn

$$(x-x_0)f_x(x_0,y_0)+(y-y_0)f_y(x_0,y_0)-(z-z_0)=0$$

or equivalently $z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$.

or other forms

Ex: The surface $z = 2x^2+y^2$ is an elliptic paraboloid.

What is the equation of its tangent plane @ $(1,1,3)$?

Ex: What is the tangent plane to the sphere

$x^2+y^2+z^2 = 14$ @ the point $\langle 1,2,3\rangle$?

(Three ideas: · solve for z · differentiate implicitly · geometry.)

Linear approximations & differentials

Tangent plane approximation: at (x_0,y_0), plane is

$$z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

Call the RHS $L(x,y)$. This function is the linearization of f @ (x_0,y_0) — the best linear approximation

$$f(x,y) \approx L(x,y) \quad \text{when } (x,y) \text{ near}$$

(x_0,y_0)

If the tangent plane exists, this means (the $\approx$ for (x,y) near (x_0,y_0))

that $\Delta z = f(x,y) - f(x_0,y_0)$

$$= f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)$$

is given by

$$\Delta x = f_x(x_0,y_0)\,\Delta x + f_y(x_0,y_0)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x,\Delta y) \to (0,0)$

& in this case f is called <u>differentiable.</u>

When f is differentiable, one may write Δx, Δy as dx, dy & have

$$dz = f_x(x_0,y_0)\,dx + f_y(x_0,y_0)\,dy,$$

i.e. $$dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy.$$ [Total differential]

<u>Note</u>: differentiable $\Rightarrow$ continuous.

Our earlier statements are now

<u>Thm</u> If f_x & f_y exist near (x_0, y_0) & are continuous @ (x_0, y_0) Then f is differentiable @ (x_0, y_0).

<u>E.g</u>: Show $f(x,y) = xe^{xy}$ diff'ble @ $(1,0)$; linearize; & approximate $f(1.05, -0.10)$.

<u>E.g</u> For $z = f(x,y) = x^2 + 3xy - y^2$, find dz. ~~Compute~~

Gen to 3 or more vars:

$$w = g(x,y,z) \Rightarrow \Delta w = f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x,y,z)$$

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz \text{ etc.}$$