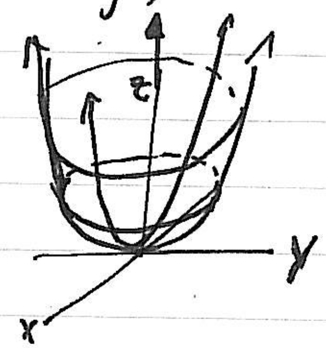
Functions of several variables

$z = f(x,y)$ $\qquad$ $q = f(x,y,z,t)$ $\qquad$ etc.

The graph of the equation $z = f(x,y)$ is the set of points $(x, y, f(x,y))$ for (x,y) in domain of f

e.g. $f(x,y) = x^2 + y^2$ $\rightsquigarrow$



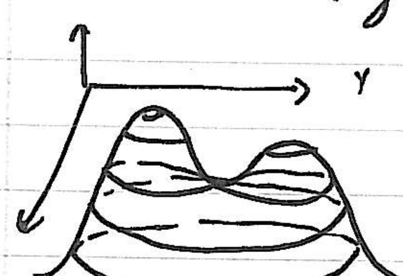
Hard to draw! One solution:
calculator or computer
(athena terminals should have Mathematica, Maple, Matlab, etc.)

2^{nd} solution: those models in the foyer

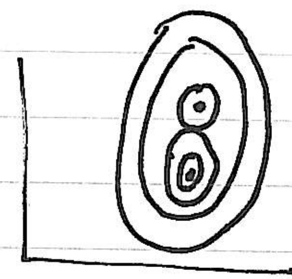
3^{rd} solution: contour plots: 2D images intended to capture many details of a 3D surface

(topographical maps)

Idea: intersect the surface with planes parallel to the xy-plane (those w/ eqns $z = \text{const.}$); these intersections are some kind of curve; draw them together



$\longrightarrow$



(could label lines to indicate heights)

Eg Level curves for $f(x,y)=xy$?

$g(x,y) = \sin x \sin y$?

Partial derivatives: idea: interested in change w.r.t. just one variable at a time.

If $f(x,y)$ is a fn then

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}, \quad \frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

Notations: for $z=f(x,y)$,

$$\frac{\partial z}{\partial x}, \ z_x, \ \frac{\partial f}{\partial x}, \ f_x, \ f_x(x,y)$$

For functions given by formulas, partial differentiation is easy: treat all other variables as constants & proceed as usual.

E.g. Compute all partial derivatives of

- $f(x,y) = x^3 - 3x^2y^3 + y^2$
- $g(x,y,t) = xe^t + \sin y \cdot \cos t$

Compute $h_x(2,1)$ if $h(x,y) = xy^2 + x^3$.

Two technical observations about how life is different in many variables than in one variable:

<u>First</u>: in one variable, we have that $\frac{df}{dx}$ represents some limit, but can also be meaningfully treated as a ratio of differentials (infinitesimals). Thus, for example, $\frac{dy}{dx} \cdot \frac{dx}{dy} = 1$. This <u>fails</u> in many variables: a partial derivative <u>cannot</u> be thought of as a ratio. For example, the <u>ideal gas law</u> says

$$PV = nRT$$

– here P, V, T vary, n, R constant. Thus

$$P = \frac{nRT}{V}, \quad V = \frac{nRT}{P}, \quad T = \frac{PV}{nR}$$

$$\Rightarrow \frac{\partial P}{\partial V} = -\frac{nRT}{V^2} \quad \frac{\partial V}{\partial T} = \frac{nR}{P}, \quad \frac{\partial T}{\partial P} = \frac{V}{nR}$$

$$\Rightarrow \frac{\partial P}{\partial V} \cdot \frac{\partial V}{\partial T} \cdot \frac{\partial T}{\partial P} = -\frac{nRT \cdot nR \cdot V}{V^2 \cdot P \cdot nR} = -\frac{nRT}{PV} = -1$$

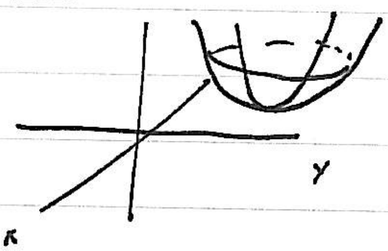
Since RHS $\neq 1$, shows "canceling" cannot be legal.

<u>Second</u>: In functions of 1 variable, having a derivative at a point means continuous at that point, i.e., no break in graph. But This fails for multivariable fns: let

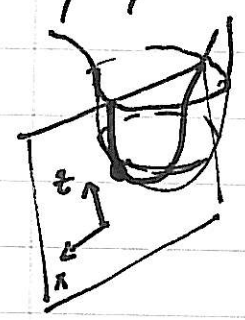
$$z = \begin{cases} 0 & \text{if } x=0 \text{ or } y=0 \\ -1 & \text{if } x \neq 0 \neq y \end{cases}$$

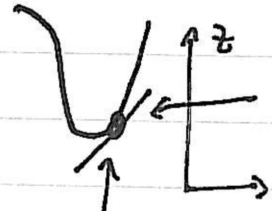
Graph:  $z_x(0,0) = z_y(0,0) = 0.$

<u>Luckily</u>: (But functions given by elementary formulas will be continuous wherever they are defined.)

Geometric interpretation: 

Surface $z = f(x,y)$.
Think about $f_x(x_0,y_0)$: cut surface with plane $y = y_0$



$\longrightarrow$ 

slope of tangent to curve of intersection

$(x_0, y_0, f(x_0,y_0))$ $\quad = f_x(x_0,y_0)$.

(&, of course, same for $\frac{\partial z}{\partial y}$: use plane $x = x_0$ instead)

Partial derivatives $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ are again fns of 2 (or more) vars $\rightsquigarrow$ may again have partial derivatives

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} f_x = f_{xx}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} f_y = f_{yx} = (f_y)_x$$

& sim. w/ x, y interchanged

(pure partials & mixed partials)

Ex: Compute all 2^{nd} partials of

$$f(x,y,z) = x^2 y + xyz$$

Funny observation: $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ & sim. for other 2 mixed cases. Coincidence? No!

Thm: If f_{xy} & f_{yx} are continuous at (x_0, y_0) & defined in a neighborhood of (x_0, y_0) then $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$

Similarly, under "nice enough" behaviour, partials of higher orders commute, e.g., should expect

$$f_{xxyz} = f_{xyxz} = f_{xyzx} = f_{xzyx} = \text{etc.}$$