18.089 Homework 6

Summer 2010

Due Friday, July 9

Question 8 updated 7/8/2010 at 10:45 AM

1. Is there a vector field $\mathbf{G}$ in space such that $\nabla \times \mathbf{F} = xy^2\mathbf{i} + yz^2\mathbf{j} + zx^2\mathbf{k}$? (Explain.)

2. Let $f = f(x, y, z)$ be a scalar function and let $\mathbf{F} = \mathbf{F}(x, y, z)$ be a vector field. Show that $\operatorname{curl}(f\mathbf{F}) = f \operatorname{curl} \mathbf{F} + (\nabla f) \times \mathbf{F}$.

3. Let $f = f(x, y, z)$ be a scalar function and let $\mathbf{F} = \mathbf{F}(x, y, z)$ be a vector field. Expand $\operatorname{div}(f\mathbf{F})$. (You should be able to write your answer in a way that resembles the product rule.)

4. What is the surface with parametric equations

$$\mathbf{r}(u, v) = \langle (2 + v\cos(u/2))\cos u, (2 + v\cos(u/2))\sin u, v\sin(u/2)\rangle$$

   for $0 \leq u \leq 2\pi$ and $-1 \leq v \leq 1$? (Intended solution: find an adequate CAS. Plot the surface. Recognize it. Write the name of the surface here. No need to attach images or explanation.)

5. The surface S parametrized by $\mathbf{r}(u, v) = \langle u\cos v, u\sin v, v\langle$ for $0 \leq u \leq 1$ and $0 \leq v \leq \pi$ is part of a *helicoid*. Compute $\iint_S \sqrt{1 + x^2 + y^2}$ dSA.

6. Use Stokes' Theorem to evaluate the integral $\iint_S \operatorname{curl} \mathbf{F} \cdot \hat{\mathbf{n}}$ dSA, where

   (a) $\mathbf{F} = \langle x^2e^{yz}, y^2e^{xz}, z^2e^{xy}\rangle$ and S is the hemisphere $z = \sqrt{4 - x^2 - y^2}$, oriented upwards.

   (b) $\mathbf{F} = \langle xyz, xy, x^2yz\rangle$ and S consists of the four sides and the top (but not the bottom) of the cube with vertices $(\pm 1, \pm 1, \pm 1)$, oriented outwards. (Hint: it may be helpful to choose some other surface with the same boundary.)

7. Use Stokes' Theorem to evaluate the integral $\oint_C \mathbf{F} \cdot d\mathbf{R}$, where

   (a) $\mathbf{F} = \langle x + y^2, y + z^2, z + x^2\rangle$ and C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, oriented counterclockwise when viewed from above.

(b) $\mathbf{F} = \langle 2z, 4x, 5y \rangle$ and C is the intersection of the plane $z = x + 4$ with the cylinder $x^2 + y^2 = 4$, oriented counterclockwise when viewed from above.

8. Use the Divergence Theorem to evaluate the integral $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}$ dSA, where

   (a) $\mathbf{F} = \langle x^2, -2xy, xyz^2 \rangle$ and S is the surface of the solid bounded by $z = 0$ and $z = \sqrt{1 - x^2 - y^2}$.

   (b) $\mathbf{F} = \langle x, y^2, z \rangle$ and S is the surface of the tetrahedron bounded by the coordinate planes and the plane $2x + 2y + z = 6$.