18.089 Homework 3

Summer 2010

Due Tuesday, June 22

- 1. The .jpg image file available from the course website has diagrams of the graphs and contour plots for the following six functions. Match each function to its corresponding graphs. (You should try to do this *without* using a graphing utility to help!)
 - (a) $z = \sin \sqrt{x^2 + y^2}$ (b) $z = x^2 y^2 e^{-x^2 - y^2}$ (c) $z = \frac{1}{x^2 + 4y^2}$ (d) $z = x^3 - 3xy^2$ (e) $z = \sin x \sin y$ (f) $z = \sin^2 x + \frac{y^2}{4}$
- 2. For each of the following functions, compute the gradient, evaluate it at the point P, and compute the directional derivative at P in the direction \mathbf{v} .

(a)
$$f(x,y) = x^2y + xy^2$$
, $P = (-1,2)$, $\mathbf{v} = \langle 3, -4 \rangle$
(b) $g(x,y,z) = \sqrt{x^2 + y^2 + z^2}$, $P = (2,6,-3)$, $\mathbf{v} = \langle 1,1,1 \rangle$
(c) $h(w,x,y,z) = wx + wy + wz + xy + xz + yz$, $P = (2,0,-1,-1)$, $\mathbf{v} = \langle 1,-1,1,-1 \rangle$

- 3. Suppose that z = f(x, y), $x = r \cos \theta$ and $y = r \sin \theta$. Compute $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$ and show that $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2$.
- 4. A function f(x,t) is said to satisfy the (1-dimensional) wave equation if $a^2 \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial t^2}$. Which of the following functions satisfy the wave equation?

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- (a) $f(x,t) = \sin(x+at)$ (b) $f(x,t) = (x-at)^5$ (c) $f(x,t) = x^4 + 2a^2x^2t^2 + a^4t^4$ (d) $f(x,t) = x^2 + a^2t^2$
- 5. Compute the tangent plane to the following surfaces at the points indicated.
 - (a) $z = \sqrt{x^2 + y^2}$ at the point (-3, -4, 5). (Not part of the question, but something you should do: what sort of surface is this?)
 - (b) $z = e^y \cos x$ at the point (0, 0, 1)
 - (c) $xy^2 + yz^2 + zx^2 = 25$ at the point (1, 2, 3)
- 6. Find all critical points and classify by the second derivative test:

(a)
$$z = 5x^2 - 3xy + y^2 - 15x - y$$

- (b) $z = x^2 + y^3 6xy$
- 7. Let f(s,t) be the square of the distance between the points (-2 + 4s, 3 + s, -1 + 5s) and (-1 2t, 3t, 3 + t). Show that f(s,t) has a unique critical point, and that this point is a minimum. (This is one possible method to compute the distance between two skew lines.)
- 8. Consider the function $z = (y x^2)(y 2x^2)$.
 - (a) Find the unique critical point P of this function.
 - (b) Show that the second derivative test is inconclusive at this point.
 - (c) The value of the function at P is 0. Show by direct examination that there are points near P at which the function takes positive values and points near P at which the function takes negative values; this demonstrates that P is a saddle point.
 - (d) One might think that, for a critical point $P_0 = (x_0, y_0)$ of z = f(x, y), if every vertical section of the surface through P_0 has a minimum at P_0 then P_0 is a minimum point. Show that this is false by looking at the vertical sections of $z = (y - x^2)(y - 2x^2)$. (Hint: a vertical section through the origin is a plane with equation y = mx.)
- 9. For fixed a and b, the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is centered at the origin and has area πab . Using Lagrange multipliers, find the values of a and b so that the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through the point (4, 1) and has the smallest area among all such ellipses. (That is, minimize the area subject to the constraint that it passes through the point (4, 1).)

Totally optional, more challenging version of this problem: minimize the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ given that it passes through the point (1, 2, 3).

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