Quick answers to some homework questions

Joel Brewster Lewis

July 9, 2009

92. Use Green's Theorem to find the simple closed curve C (with positive orientation) for which

$$\oint_C (yx^2 + 3x)\,dx + (2xy - xy^2)\,dy$$

has the maximum value. (Hint: once you convert to a double integral, note that the integrand takes positive values in some places and negative values in other places. In order to make the integral as large as possible, what region should C enclose?)

Quick solution. By Green's, this line integral is equal to the double integral over the interior of C of

$$\begin{aligned}\frac{\partial}{\partial x}(2xy - xy^2) - \frac{\partial}{\partial y}(yx^2 + 3x) &= 2y - y^2 - x^2 \\ &= 1 - x^2 - (y-1)^2.\end{aligned}$$

This integral will be maximal when it goes only over the region in which the integrand is positive, which is precisely the interior of the circle $x^2 + (y-1)^2 = 1$. Thus, this is the desired curve. (The value of the integral, which you were not asked to compute, is $\frac{\pi}{2}$.) □

93. Let $\vec{F} = \langle 2xy(1-x^2), y^2(3x^2-1)\rangle$. For each of the following curves, find the flux of $\vec{F}$ across the curve.

   (a) The line segment joining $(-1,1)$ to $(1,1)$.
   (b) The lower half of the unit circle $y = -\sqrt{1-x^2}$ joining $(-1,0)$ to $(1,0)$.

Quick solution. The answer to both is 0. For the first, we just have to compute $\int_{-1}^{1} 2x \cdot 1 \cdot (1-x^2)\,dx$. For the second, taking the parametrization $x = \cos t$, $y = \sin t$ for $\pi \le t \le 2\pi$ leads (with some manipulations) to an integrand of $2\sin^2 t \cos t - 5\sin^4 t \cos t$, and this can be integrated with a trig substitution. □

94. Show that the vector field $\vec{F}(x,y) = \langle ye^{xy} + 2, xe^{xy} - 1\rangle$ is conservative, and find a scalar field $f(x,y)$ such that $\vec{\nabla} f = \vec{F}$.

Quick solution. Let the coordinates of $\vec{F}$ be M and N as usual. We have

$$\frac{\partial M}{\partial y} = e^{xy} + xye^{xy} = \frac{\partial N}{\partial x},$$

so $\vec{F}$ is conservative. Use either method introduced in class to find $f(x,y) = e^{xy} + 2x - y$. □

95. Which of the following three-dimensional vector fields are conservative? (You *don't* need to compute the associated scalar fields.)

(a) $\langle y + 2xz, x + z^2, x^2 + 2yz\rangle$

(b) $\langle 1 + y + yz, x + xz, xy + y\rangle$

Quick solution. We can compute the curl of the two vector fields (which, as we saw, is equivalent to comparing the appropriate partial derivatives of the components); the curl of the first is $\langle 2z - 2z, 2x - 2x, 1 - 1\rangle = \vec{0}$ and the curl of the second is $\langle (x+1) - x, y - y, (1 + z) - (1 + z)\rangle \neq \vec{0}$, so the first is conservative and the second isn't.

(The scalar field associated to the first, which you weren't asked to find, is $xy + yz^2 + x^2 z$.) □

96. Compute the divergence and curl of the following vector fields.

(a) $\vec{F} = \langle 2x^2 y, 3xz^3, xy^2 z^2\rangle$

(b) $\vec{G} = e^{x+y}\,\hat{\imath} + e^z \cos y\,\hat{\jmath} + e^z \sin y\,\hat{k}$

(c) $\vec{H} = \langle 3, x + y + z, x^2 + y^2 + z^2\rangle$

Quick solution. $\operatorname{div}\vec{F} = 4xy + 0 + 2xy^2 z = 4xy + 2xy^2 z$.

$\operatorname{div}\vec{G} = e^{x+y} - \sin y e^z + e^z \sin y = e^{x+y}$.

$\operatorname{div}\vec{H} = 0 + 1 + 2z = 2z + 1$.

$\operatorname{curl}\vec{F} = \langle 2xyz^2 - 9xz^2, -y^2 z^2, -2x^2 + 3z^3\rangle$.

$\operatorname{curl}\vec{G} = \langle 0, 0, -e^{x+y}\rangle$

$\operatorname{curl}\vec{H} = \langle 2y - 1, -2x, 1\rangle$ □

97. Use the Divergence Theorem to find the flux of the given vector field over the given surface S.

(a) $\vec{F} = \langle x, y, z\rangle$ and S is the surface of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

(b) $\vec{F} = \langle xz, y^2, x\rangle$ and S is the surface of the tetrahedron whose vertices are $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 2)$. (It may be helpful to note that the faces of this tetrahedron are the coordinate planes together with the plane $2x + 2y - z = 0$.)

(c) $\vec{F} = \langle x, -y, z\rangle$, and S is the cylinder whose curved surface has equation $x^2 + y^2 = r^2$ and whose bases have equations $z = 0$ and $z = b$.

Quick solution. The divergence theorem says that the flux $\iint_S \vec{F} \cdot \hat{n}\, dA$ is equal to the triple integral $\iiint_R \operatorname{div}\vec{F}\, dV$. For part (a), this gives the desired value to be $\iiint_R 3\, dV = 3\,\mathrm{Vol}(R) = 4\pi abc$.

The second question contains an error: the plane in question is actually $2x + 2y + z = 2$. With this correction, we have that the desired flux is

$$\int_0^1 \int_0^{1-x} \int_0^{2-2x-2y} z + 2y\, dz\, dy\, dx = \frac{1}{3}.$$

Finally, for the third part, we have that the desired flux is $\iiint_R 1\, dV = \mathrm{Vol}(R) = \pi r^2 b$. □