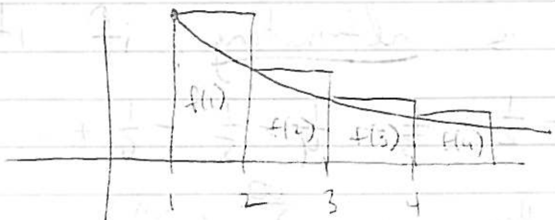
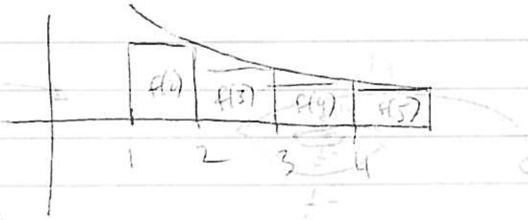
Back to infinite series: ~~three~~ four important tests.

① Integral test



Suppose f is a decreasing, positive function.



Then

$$\sum_{n=1}^{\infty} f(n) \geq \int_{1}^{\infty} f(x)\,dx \geq \sum_{n=2}^{\infty} f(n)$$

(& can shift if necessary). Note $\sum_{n=1}^{\infty} f(n)$ & $\sum_{n=2}^{\infty} f(n)$ just differ by a constant (namely, $f(1)$), so they behave similarly (both converge or both diverge). Since the integral is between them, we deduce

$$\int_{1}^{\infty} f(x)\,dx \text{ converges if and only if } \sum_{n=1}^{\infty} f(n)$$

converges (for positive, decreasing functions f).

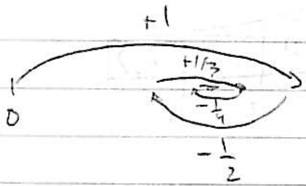
Ex • For what values of p does $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge?

• Does $\sum_{n=3}^{\infty} \frac{1}{n \ln n}$ converge? What about $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$?

Note: also lets us estimate integrals by sums & vice-versa.

(2) Alternating series test

A series is *alternating* if its terms alternate signs, e.g. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots$, i.e., it can be written as $\sum_{n=0}^{\infty}(-1)^n a_n$ where $a_n>0$ for all n.



← if the terms a_n (i.e., the positive part of the original series) are decreasing, something like this happens.

Thus, *Alternating series test*: If a_n is a sequence of positive numbers that decrease to zero then

$$\sum_{n=0}^{\infty}(-1)^n a_n \quad \text{converges.}$$

(If terms don't decrease to 0, we know nothing.)

E.g: Which of the following converge?

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\cdots$$

$$1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{5}}-\cdots$$

$$\sum_{n=1}^{\infty}(-1)^n\frac{n}{n+1}$$

③ Root test: (comparison w/ geometric series)

If a_n is any sequence and

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1 \quad \text{then}$$

$$\sum_{n=0}^{\infty} a_n \quad \text{converges.}$$

If

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1 \quad \text{then}$$

$$\sum_{n=0}^{\infty} a_n \quad \text{diverges.}$$

Otherwise, don't know.

E.g.:

$$\sum_{n=0}^{\infty} x^n$$

$$\sum_{n=0}^{\infty} \frac{17^n}{n^n}$$

④ Ratio test (also comparison w/ geometric series)

If a_n is any sequence &

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = L < 1 \quad \text{then}$$

$$\sum_{n=0}^{\infty} a_n \quad \text{converges}$$

$$'' \quad = L > 1 \quad \text{then}$$

$$'' \quad \text{diverges.}$$

O/W, don't know.

E.g. • $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

• $\sum_{n=1}^{\infty} \frac{10^n}{(n+1)\cdot 4^{2n+1}}$

• $1 + \frac{1\cdot 2}{1\cdot 3} + \frac{1\cdot 2\cdot 3}{1\cdot 3\cdot 5} + \frac{1\cdot 2\cdot 3\cdot 4}{1\cdot 3\cdot 5\cdot 7} + \cdots + \frac{n!}{1\cdot 3\cdot 5\cdot 7\cdot \ldots \cdot (2n-1)} + \cdots$

• $\sum_{n=0}^{\infty} \frac{(2n)!}{n!\cdot n!\cdot 3^n}$

Power series

We have seen that $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ provided $|x| < 1$. Reinterpret this: the function $T(x) = \frac{1}{1-x}$ is the sum of the infinite series involving the variable x. When else can we do this, i.e., write

$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$ (an "infinite polynomial")?

where the a_n are constants and x is a variable, or what are the properties of a function given by such an expression?

Note: Every power series converges at 0. Some converge only at 0, e.g., $\sum_{n=1}^{\infty} n^n x^n = x + 4x^2 + 27x^3 + 256x^4 + \cdots$

Every power series exhibits exactly one of three

behaviors: (i) converges *only* at $x=0$
(ii) converges for all real x (e.g., $\sum_{n\geq 1} \frac{x^n}{n^n}$)
(iii) There is some positive number R s.t.

$|x|<R \Rightarrow$ converges, $|x|>R \Rightarrow$ diverges,
& we don't know what happens at $x=\pm R$.

(This essentially follows from ratio &/or root tests.)

How to find R: $R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$ (if exists)

Ex: radius of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$?

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$?

When $|x|<R$, $f(x) = \sum_{n\geq 0} a_n x^n$ behaves just like a polynomial (which is a very special kind of power series). Specifically, it's continuous, differentiable, its derivative is

$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$

& $\int f(x)\,dx = a_0 x + \frac{a_1 x^2}{2} + \frac{a_2 x^3}{3} + \frac{a_3 x^4}{4} + \cdots$

This means $a_1 = f'(0)$, $2a_2 = f''(0)$, $6a_3 = f'''(0)$, ...

... $n!\, a_n = f^{(n)}(0)$, ...

i.e., $\boxed{f(x) = f(0) + \frac{f'(0)}{1!} x^1 + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots}$

Important special cases

(*) $e^x =$

(*) $\sin x =$

(*) $\cos x =$

Note that power series behave "like they should," so to get power series for e^{2x}, just put $2x$ in to the power series for e^x

$e^{2x} =$

(*) $e^{-x} =$

$\cosh x =$

$\sinh x =$

(*) $\ln(1+x) =$

(*) $\frac{1}{1-x} =$

Now how about $\frac{1}{(1-x)^2}$? $\frac{1-\sqrt{1-4x}}{2}$?