Announce existence of midterm solutions

Matrices :

A matrix is just a $\underline{\text{rectangular}}$ table of #s

$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & 3 \\ 0 & 4 & 5 \end{pmatrix}$$

↑ Mostly interested in $\underline{\text{square}}$ matrices.

Matrices can be used to summarize information, e.g.

Shop sells ice cream, froyo & sorbet

cost per pint

	i.c.	f.y.	s.
labor	.20	.30	.25
ingredients	.40	.40	.30
mktg	.10	.20	.20

input vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ (vector as matrix)

output vector $= \vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ = cost vector

So can find $\vec{c}$ in terms of $\vec{x}$ & can reverse this process by back-solving. To abbreviate the system of eqns, can write

$$\begin{bmatrix} .2 & .3 & .25 \\ .4 & .4 & .3 \\ .1 & .2 & .2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1^{st} \text{ row} \cdot \vec{x} \\ 2^{nd} \text{ row} \cdot \vec{x} \\ 3^{rd} \text{ row} \cdot \vec{x} \end{bmatrix}$$

↑ dot product of vectors

& this defines the product of a matrix w/ a column vector. Can extend this to multiply any matrices $A \cdot B$ where # cols A = # Rows B :

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$$\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}\begin{pmatrix}1&-1\\0&1\\2&-1\end{pmatrix} = \begin{matrix} & \begin{pmatrix}1&-1\\0&1\\2&-1\end{pmatrix}\\ \begin{pmatrix}1&2&3\\4&5&6\end{pmatrix} & \begin{pmatrix}7&-2\\16&-5\end{pmatrix}\end{matrix}$$

$$A \cdot B = A \cdot [\underbrace{\vec{b}_1, \vec{b}_2, \ldots}_{\text{column vectors}}] = [A\vec{b}_1, A\cdot\vec{b}_2, \ldots] = [c_{ij}]$$

where $c_{ij} = (i^{th} \text{ row of } A) \cdot (j^{th} \text{ column of } B)$

Also have addition of same-shape matrices & scalar multiplication.

The transpose A^T of A is the matrix we get flipping A across its diagonal: $\begin{bmatrix}1&1&0\\1&2&3\end{bmatrix}^T = \begin{bmatrix}1&1\\1&2\\0&3\end{bmatrix}$.

The $(n \times n)$ Identity matrix I_n is $\begin{pmatrix}1&&&0\\&1&&\\&&\ddots&\\0&&&1\end{pmatrix}$.

Note that for any $n \times m$ matrix M (including possibly a $n \times 1$ column vector) we have $I_n \cdot M = M$, & similarly for any $m \times n$ matrix N (including possibly a $1 \times n$ row vector) we have $N \cdot I_n = N$.

(So for any 3×3 matrix A, $I_3 \cdot A = A \cdot I_3 = A$.)

Also for square matrices $\det(A \cdot B) = \det(A) \cdot \det(B)$:

e.g. $\begin{bmatrix}2&1\\0&1\end{bmatrix} \cdot \begin{bmatrix}1&0\\-1&-1\end{bmatrix} = \begin{bmatrix}1&-1\\-1&-1\end{bmatrix}$

$\det\begin{bmatrix}2&1\\0&1\end{bmatrix} = 2$; $\det\begin{bmatrix}1&0\\-1&-1\end{bmatrix} = -1$; $\det\begin{bmatrix}1&-1\\-1&-1\end{bmatrix} = -2$.

Given a square (say, 3x3) matrix A s.t. $\det(A) \neq 0$, there is a unique matrix A^{-1} s.t. $A \cdot A^{-1} = I_3 = A^{-1} \cdot A$.

Observe that $\det(A^{-1}) = \dfrac{1}{\det(A)}$

How to compute the inverse: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Step 1: Compute $\det A$. If $\det A = 0$, give up (no inverse exists)

Step 2: Calculate the minors

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

where M_{ij} is what we get when we cover up row i & column j & take the determinant.

Step 3: Multiply by the checkerboard $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

- The result is the matrix of cofactors.

Step 4: Take the transpose. The result is the adjoint A^* of A.

Step 5: $A^{-1} = \dfrac{1}{\det A} \cdot A^*$

Obs: This works for matrices of any size.

E.g: what's the inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$?

(Check that their dets are inverses.)

For example, let $A = 20\begin{bmatrix} 0.2 & .3 & .25 \\ .4 & .4 & .3 \\ .1 & .2 & .2 \end{bmatrix} = \begin{bmatrix} 4 & 6 & 5 \\ 8 & 8 & 6 \\ 2 & 4 & 4 \end{bmatrix}$

Minors are $\begin{bmatrix} 8 & 20 & 16 \\ 4 & 6 & 4 \\ -4 & -16 & -16 \end{bmatrix} \longrightarrow \begin{bmatrix} 8 & -20 & 16 \\ -4 & 6 & -4 \\ -4 & 16 & -16 \end{bmatrix} \longrightarrow \begin{bmatrix} 8 & -4 & -4 \\ -20 & 6 & 16 \\ 16 & -4 & -16 \end{bmatrix}$

$$\det A = (4\cdot 8\cdot 4 + 6\cdot 6\cdot 2 + 5\cdot 8\cdot 4 - 2\cdot 8\cdot 5 - 4\cdot 6\cdot 4 - 4\cdot 8\cdot 6)$$

$$= -8$$

So $A^{-1} = \begin{bmatrix} -1 & \frac{1}{2} & -\frac{1}{2} \\ 5/2 & -3/4 & -2 \\ -2 & 1/2 & 2 \end{bmatrix}$

& to get the inverse of our original matrix, multiply by 20.

So what? Well, if one day costs were $\bar{c} = \begin{bmatrix} 100 \\ 40 \\ \cancel{20} \end{bmatrix}$ 80 120 52

Then parts purchased is

$$20\begin{bmatrix} -1 & \frac{1}{2} & -\frac{1}{2} \\ 5/2 & -3/4 & -2 \\ -2 & 1/2 & 2 \end{bmatrix}\begin{bmatrix} 80 \\ 120 \\ 52 \end{bmatrix} = \begin{bmatrix} 120 \\ 120 \\ 80 \end{bmatrix}.$$