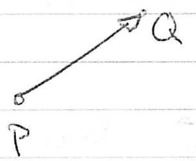
Detour requested to do some linear algebra stuff ASAP; we'll return to finish sequences & series afterwards.

Basic concepts: geometrically, a vector is just a directed line segment (2 or 3 dims doesn't matter)



$\overrightarrow{PQ}$. Has length, $|\overrightarrow{PQ}|$.

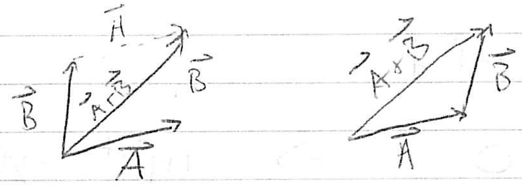
If $|\overrightarrow{PQ}| = 1$, say we have a <u>unit vector</u>

Has direction (given by associated unit vector) (except zero vector)

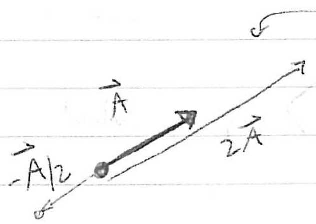
$$\operatorname{dir}(\vec{A}) = \frac{\vec{A}}{|\vec{A}|} = \vec{A} \cdot \frac{1}{|\vec{A}|}.$$

<u>Note</u>: vectors are the same if they only differ in location. i.e., if they are parallel & have the same length & direction.

Can add vectors:



Can multiply vectors by a <u>constant to scale</u> them, ← <u>scalar</u> multiplication



Can also look algebraically:

If $P = (p_1, p_2, p_3)$ & $Q = (q_1, q_2, q_3)$

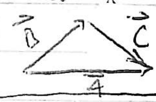
then $\overrightarrow{PQ} = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle$.

Then addition & scalar multiplication are "componentwise," i.e. $\langle a_1, \dots \rangle + \langle b_1, \dots \rangle = \langle a_1 + b_1, \dots \rangle$ & $r \cdot \langle a_1, \dots \rangle = \langle ra_1, \dots \rangle$.

We can also define a product between vectors:

dot product: $\vec{A} \cdot \vec{B} = \begin{cases} |\vec{A}| \cdot |\vec{B}| \cdot \cos\theta \\ \sum a_i b_i \end{cases}$ ← angle between $\vec{A}$ & $\vec{B}$

(Proof of equivalence via Law of Cosines in



)

Ex: $\langle 0,0,1 \rangle \cdot \langle 1,1,0 \rangle = 0$

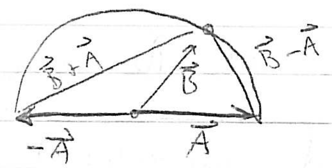
More generally, $\vec{A} \cdot \vec{B} = 0 \iff \vec{A}$ & $\vec{B}$ are orthogonal (perpendicular) or one of them is $\vec{0}$.

Also $\vec{A} \cdot \vec{A} = |\vec{A}|^2 = a_1^2 + a_2^2 + a_3^2$

In addition, $\vec{A} \cdot \hat{u}$ is the component of $\vec{A}$ in the $\hat{u}$ direction, where $\hat{u}$ denotes a unit vector. (Esp: $\hat{i} = \langle 1,0,0 \rangle$, $\hat{j} = \langle 0,1,0 \rangle$, $\hat{k} = \langle 0,0,1 \rangle$ are coordinate unit vectors.)

One nice use of vectors:

semi-circle.



$$(\vec{B}+\vec{A}) \cdot (\vec{B}-\vec{A}) = \vec{B}\cdot\vec{B} - \vec{B}\cdot\vec{A} + \vec{A}\cdot\vec{B} - \vec{A}\cdot\vec{A}$$

$$= |\vec{B}|^2 - |\vec{A}|^2 = 0 \implies \text{right angle.}$$

If $\vec{A} = \langle 2,1,2 \rangle$ & $\vec{B} = \langle 2,2,-1 \rangle$, what is the angle between $\vec{A}$ & $\vec{B}$?

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\cdot|\vec{B}|} = \frac{4+2-2}{\sqrt{4+1+4}\cdot\sqrt{4+1+4}} = \boxed{\frac{4}{9}}$$

$$\theta = \arccos(4/9).$$

Determinants

Given a square table of #s, the determinant associates a value as follows:

$$|a| = a \qquad \text{(easy)}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad \left(= a|d| - b|c| \right)$$

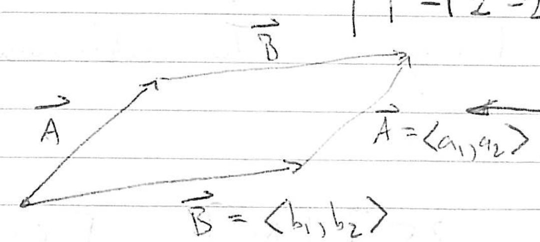
← determinants, NOT absolute values

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

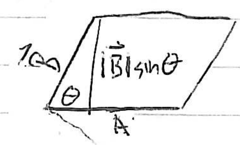
(Can instead expand by any row or column, just remember the checkerboard rule for signs: $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$, $\begin{vmatrix} + & - \\ - & + \end{vmatrix}$)

Ex: Compute $\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix}$, $\begin{vmatrix} 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 1 & -1 & 2 & -2 \end{vmatrix}$, $\begin{vmatrix} 2 & 1 & 3 \\ 1 & 1 & 0 \\ 2 & 5 & 1 \end{vmatrix}$

So what? Well,



← what's the area of this parallelogram?



Area = base · height

$= |\vec{A}|\,|\vec{B}| \sin\theta$

$= |\vec{A}| \cdot |\vec{B}'| \cdot \cos(\pi/2 - \theta)$

$= \vec{A} \cdot \vec{B}'$

$= \langle a_1, a_2 \rangle \cdot \langle b_2, -b_1 \rangle$

$\vec{B}' = \vec{B}$ rotated 90°

$= \langle b_2, -b_1 \rangle$

$= a_1 b_2 - a_2 b_1 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} \vec{A} \\ \vec{B} \end{vmatrix}.$

Similarly, 3 vectors define a parallelepiped whose volume is $\pm \begin{vmatrix} \vec{A} \\ \vec{B} \\ \vec{C} \end{vmatrix}$.

Following 18

Quicker method to compute 3×3 determinants:

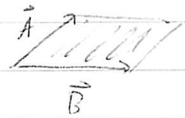
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \left|\begin{array}{ccc|cc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array}\right.$$

<u>ONLY</u> 2×2 & 3×3 !

A second vector product: The "cross product" or "vector product"

$\boxed{\text{ONLY in 3 dimensions}}$

Geometrically, $\vec{A}\times\vec{B}$ is a vector whose length is

The area of 

$= |\vec{A}|\,|\vec{B}|\sin\theta$ & is perpendicular

to the plane defined by $\vec{A}$ & $\vec{B}$, using right-hand rule

(so $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$.)

Analytically, $\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}$

<u>Note</u>: • $\vec{A}\times\vec{A} = \vec{0}$; in general, cross product of any two parallel vectors is $\vec{0}$.

• $\vec{A}\times(\vec{B}\times\vec{C}) \neq (\vec{A}\times\vec{B})\times\vec{C}$.

Compute • $(2\hat{i}+\hat{j})\times(-\hat{i}+\hat{j}+\hat{k}) = ?$

• $((\hat{i}+\hat{j})\times\hat{i})\times\hat{j} = ?$

• $(\hat{i}+\hat{j})\times(\hat{i}\times\hat{j}) = ?$

Matrices :

A matrix is just a rectangular table of #s

↑ Mostly interested in square matrices.

$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & 3 \\ 0 & 4 & 5 \end{pmatrix}$$

Matrices can be used to summarize information, e.g.

Shop sells ice cream, froyo & sorbet

cost per pint

	i.c.	f.y.	s.
labor	.20	.30	.25
ingredients	.40	.40	.30
mktg	.10	.20	.20

input vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ (vector as matrix)

output vector $= \vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ = cost vector

So can find $\vec{c}$ in terms of $\vec{x}$ & can reverse this process by back-solving. To abbreviate the system of eqns, can write

$$\begin{bmatrix} .2 & .3 & .25 \\ .4 & .4 & .3 \\ .1 & .2 & .2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1^{st} \text{ row} \cdot \vec{x} \\ 2^{nd} \text{ row} \cdot \vec{x} \\ 3^{rd} \text{ row} \cdot \vec{x} \end{bmatrix}$$

↑ dot product of vectors

& this defines the product of a matrix w/ a column vector. Can extend this to multiply any matrices $A \cdot B$ where # cols A = # Rows B :