In our case,

$$p \neq 1 \Rightarrow \int_1^a \frac{dx}{x^p} = \frac{x^{-p+1}}{-p+1}\Bigg|_1^a = \frac{a^{-p+1}}{-p+1} + \frac{1}{p-1}$$

If $p < 1$, $\quad a^{-p+1} \xrightarrow{a\to\infty} \infty \quad \Rightarrow$ diverges.

If $p > 1$, $\quad a^{-p+1} \xrightarrow{a\to\infty} 0 \quad \Rightarrow$ converges (to $\frac{1}{p-1}$).

If $p = 1$, $\quad \displaystyle\int_1^a \frac{dx}{x} = \ln(a) - \ln(1) = \ln(a) \longrightarrow \infty$, diverges.

Other examples:

- $\displaystyle\int_0^\infty e^{-x}\,dx = e^{-x}\Big|_0^\infty = 1 - \underset{0}{\underset{\shortparallel}{e^{-\infty}}} = \boxed{1}$

- $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \arctan x\Big|_{-\infty}^\infty = \pi/2 - (-\pi/2) = \pi$

- $\displaystyle\int_e^\infty \frac{dx}{x\ln x} =$

- $\displaystyle\int_e^\infty \frac{dx}{x(\ln x)^2} =$

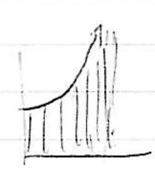
Can also have improper integrals that are "infinitely high"

$$\int_0^1 \frac{1}{x^p}\,dx$$

$(u = 1/x)$

or

$$\int_0^1 \frac{dx}{\sqrt{1-x}}$$

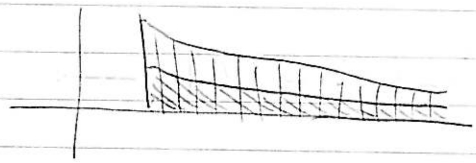


$\left(u = \frac{1}{1-x}\right)$

When do improp integrals converge? Well, some we can integrate; others, can use <u>comparison tests</u>: If $0 \leq f(x) \leq g(x)$ $\forall x$

Then $\int_a^\infty g(x)\,dx$ converges $\Rightarrow$ $\int_a^\infty f(x)\,dx$ converges

$\int_a^\infty f(x)\,dx$ diverges $\Rightarrow$ $\int_a^\infty g(x)\,dx$ diverges



<u>Limit comparison</u>: if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L \neq 0$ is finite

Then $\int_a^\infty f(x)\,dx$ has same behavior as $\int_a^\infty g(x)\,dx$

E.g.: $\displaystyle\int_1^\infty \frac{dx}{x^6+3x^3+2x^2+1}$

$\displaystyle\int_1^\infty \frac{\ln(x)}{x^3}\,dx$

<u>Infinite series</u>: $\frac{1}{2}+\frac{1}{4}=$

$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=$

$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=$

$\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n}=$

$\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n}+\cdots=$

In general, $1+x+x^2+\dots+x^n = \begin{cases} n+1, & x=1 \\ \dfrac{1-x^{n+1}}{1-x}, & x\neq 1 \end{cases}$.

Pf: $(1-x)(1+x+\dots+x^n) = 1+x+x^2+\dots+x^{n-1}+x^n$
$\quad -x-x^2-\dots-x^{n-1}-x^n-x^{n+1}$
$= 1-x^{n+1}$

We say series $a_1+a_2+\dots+a_n+\dots = \sum_{n=1}^{\infty} a_n$ is convergent if the partial sums $a_1+a_2+\dots+a_n$ approach some fixed value l. This l is the sum of the infinite series.

E.g. for $\sum_{n=0}^{\infty} x^n = 1+x+\dots$, partial sums are

$$1+\dots+x^n = \frac{1-x^{n+1}}{1-x} \xrightarrow[n\to\infty]{} \begin{cases} \frac{1}{1-x} & \text{if } |x|<1 \\ \infty & \text{if } |x|>1 \\ \infty & \text{if } x=1 \\ \text{DNE} & \text{if } x=-1 \end{cases}$$

E.g. $0.3333\dots = \frac{3}{10}+\frac{3}{100}+\frac{3}{10^3}+\dots$

$= \frac{3}{10}\left(1+\frac{1}{10}+\frac{1}{10^2}+\dots\right)$ $\quad |\frac{1}{10}|<1$, so

$= \frac{3}{10}\left(\frac{1}{1-1/10}\right) = \boxed{\frac{1}{3}}$.

IF series converges, terms a_n must go to 0. (n^{th} term test)

But the reverse statement is FALSE:

$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\dots >$

$1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\dots = 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\dots \to \infty$

So we need better tools to test for convergence.

- <u>Comparison test</u> If a_n, b_n are two <u>nonnegative</u> sequences & $a_n \le b_n$ Then

$$\sum b_n \text{ conv} \Rightarrow \sum a_n \text{ conv}$$
$$\sum a_n \text{ div} \Rightarrow \sum b_n \text{ div}$$

<u>Examples</u>: $\displaystyle\sum_{n\ge 0} \frac{1}{3^n+1}$

$$\sum_{n\ge 1} \frac{1}{\sqrt{n}}$$

- <u>Limit comparison test</u>

If $a_n \ge 0$, $b_n > 0$ & $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = L$, $0 < L < \infty$

Then $\sum a_n$ & $\sum b_n$ have same behavior.

<u>E.g.</u>, $\displaystyle\sum_{n\ge 2} \frac{n}{n^2-1}$

- $\displaystyle\sum_{n\ge 1} \frac{3n+2}{n} \cdot \frac{4^n}{5^n+1}$

- $\displaystyle\sum \frac{1}{n+\sqrt{n}}$

Another example: $\displaystyle\sum_{n\ge 1} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{10} + \dots$

$$\sum_{n=1}^{m} \frac{1}{n(n+1)} = \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \dots + \left(\frac{1}{m} - \frac{1}{m+1}\right) = 1 - \frac{1}{m+1} \longrightarrow 1$$

$\Rightarrow$ converges, so $\displaystyle\sum_{n\ge 1}^{\infty} \frac{1}{n^2}$?