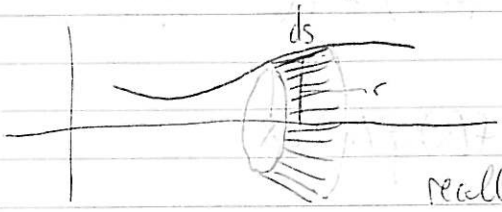
Surface areas of solids of rotation:



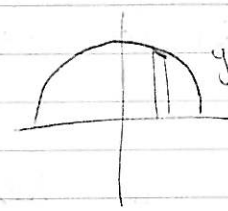
Area of this piece is $2\pi r\, ds$;

recall $ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$, & here $r = y$;

so $\text{area}_{\text{strip}} = y\sqrt{1+(y')^2}\, dx$

$$\text{total area} = \int_a^b y\sqrt{1+(y')^2}\, dx \qquad (y = f(x))$$

<u>Example</u>: Area of a sphere:



$$\frac{dy}{dx} = \frac{-x}{\sqrt{R^2-x^2}}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{x^2}{R^2-x^2} \quad \Rightarrow \quad \text{area strip} = 2\pi\sqrt{R^2-x^2}\sqrt{1+\frac{x^2}{R^2-x^2}}\, dx$$

$$= 2\pi R\, dx$$

$\Rightarrow$ surface area of whole sphere

$$= \int_{-R}^{R} 2\pi R\, dx = 2\pi R x \Big|_{-R}^{R} = 4\pi R^2$$

$$\text{alt.} = 2\int_0^R 2\pi R\, dx = 4\pi R x\Big|_0^R = 4\pi R^2.$$

Note also that this argument shows that the area of a slice of a sphere depends only on the height of the slice.

Integration by parts

Recall the product rule: $\frac{d}{dx}\left(f(x)\cdot g(x)\right) = \left(\frac{d}{dx}f(x)\right)\cdot g(x) + f(x)\cdot\frac{d}{dx}g(x)$

$$\Rightarrow \int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx$$

(sometimes written $\int u\,dv = uv - \int v\,du$:

$f(x) = v \Rightarrow dv = f'(x)dx$

$g(x) = u \Rightarrow du = g'(x)dx$)

Can use this to simplify many integrals:

$$\int x e^x\,dx = xe^x - \int e^x dx = xe^x - e^x$$

$u = x \quad dv = e^x dx$

$du = dx \quad v = e^x$

$$3\int x^2 \ln x\,dx = x^3 \ln x - \int x^3\cdot\frac{1}{x}\,dx = x^3\ln x - \frac{x^3}{3}$$

$u = \ln x \quad dv = 3x^2 dx$

$du = \frac{1}{x}dx \quad v = x^3$

Informal rules:
- logarithms want to be differentiated
- polynomials want to be differentiated, but not as much as logs
- exponential & trig functions don't care.

Note: May have to integrate by parts more than once. Also,

$$I = \int e^x \sin x\,dx = -e^x\cos x + \int e^x \cos x\,dx = -e^x\cos x + e^x \sin x - \int e^x \sin x\,dx$$

$u = e^x \quad dv = \sin x\,dx$ $\qquad$ $u = e^x \quad dv = \cos x\,dx$

$$\Rightarrow I = \frac{e^x\sin x - e^x\cos x}{2}$$

Some assorted topics: ① indeterminate forms & L'Hopital's rule

Sometimes we have limits that we don't know how to evaluate;

$\lim_{x\to\infty} \frac{e^x}{x}$ looks like $\frac{\infty}{\infty}$, & we need to know which is "bigger", i.e., how "the two infinities compare"

$$\boxed{\text{If } \lim_{x\to a} \frac{f(x)}{g(x)} \text{ is an indeterminate ratio } \left(\tfrac{0}{0} \text{ or } \tfrac{\infty}{\infty}\right), \text{ \& } \lim_{x\to a}\frac{f'(x)}{g'(x)} \text{ exists, then } \lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.}$$

So, e.g., $\lim_{x\to\infty} \frac{e^x}{x} = \lim_{x\to\infty}\frac{e^x}{1} = \infty$.

$$\lim_{x\to 0} \frac{e^x-1}{x} = \qquad\qquad \lim_{x\to 0}\frac{x^2+2x}{x^2+2} =$$

$$\lim_{x\to 0}\frac{e^x-1-x}{x^2} =$$

$$\lim_{x\to 0}\frac{x^3}{x-\sin x} =$$

· Other (non-ratio) indeterminate forms:

$$\underbrace{0\cdot\infty}_{\text{rewrite as } \frac{0}{1/\infty} \text{ or } \frac{\infty}{1/0}}, \quad \underbrace{\infty-\infty}_{\text{take the exponential}}, \quad \underbrace{0^0, \quad \infty^0, \quad 1^\infty}_{\text{take log}}$$

or do algebraic trickery in these cases.

e.g., $\lim_{x\to 0^+} x\cdot\ln x$

$\lim_{x\to\infty} \ln(e^x+x) - x$ or $\lim_{x\to\infty} \sqrt{x^2+x} - x$

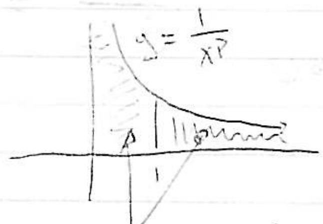
$\lim_{x\to\infty} x^{1/x}$

$\lim_{x\to 0^+} x^{\left(2/\ln(x)\right)}$

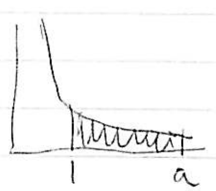
$\lim_{x\to 0} (1+ax)^{1/x}$

Lecture 5 ended here (2 hrs)

<u>Improper integrals</u>: We said we don't want to integrate across holes... what about "just up to" a hole?



are these two regions finite or infinite?

 $= \lim_{a\to\infty}$ 

. So, is

$$\lim_{a\to\infty}\int_1^a \frac{dx}{x^p} =: \underbrace{\int_1^\infty \frac{dx}{x^p}}_{\text{an improper integral – "infinitely wide."}} \quad \text{finite or infinite?}$$

If limit exists & is finite, <u>converges</u>. O/w (limit is infinite or doesn't exist), <u>diverges</u>.