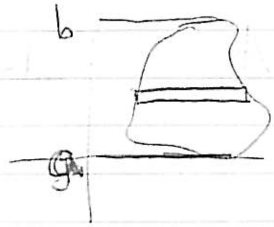
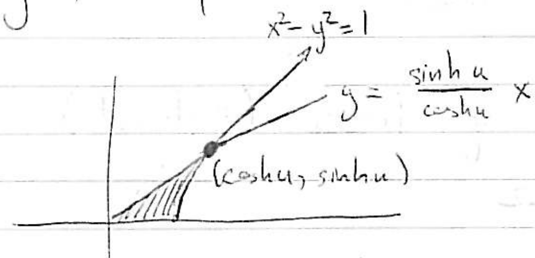
Insert
Following lecture 3.
(after p11)

Computing areas between curves by integrating w.r.t. y.



$$\int_a^b (\text{ht of rectangle})\, dy = \text{Area}.$$

E.g., problem 18:



$$A = \int_0^{\sinh u} \sqrt{y^2+1} - \frac{\cosh u}{\sinh u}\, y \; dy$$

$$= \int_0^{\sinh u} \sqrt{y^2+1}\, dy \; - \; \frac{1}{2}\cosh u \sinh u$$

$y = \sinh w$
$dy = \cosh w\, dw$
$y=0 \Rightarrow w=0$
$y=\sinh u \Rightarrow w=u$

$$\overset{\substack{y=\sinh w \\ dy=\cosh w\,dw}}{=\!=} \int_0^u \sqrt{\sinh^2 w+1}\cdot \cosh w\, dw \; - \; \frac{1}{2}\cosh u \sinh u$$

$$= \int_0^u \cosh^2 w\, dw \; - \; ''$$

$$= \frac{1}{4}\int_0^u e^{2w} + 2 + e^{-2w}\, dw \; - \; ''$$

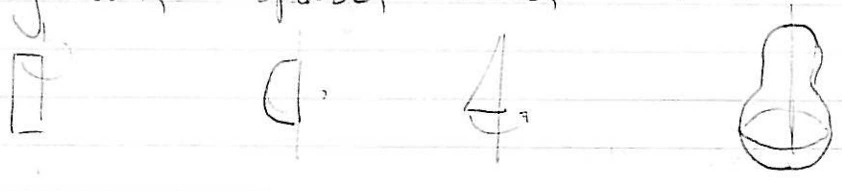
(some algebra omitted)

$$= \frac{1}{4}\left(\frac{1}{2}e^{2w} + 2w - \frac{1}{2}e^{-2w}\right)\Bigg|_{w=0}^{u} - \; '' \; = \; \boxed{\frac{1}{2}u}$$

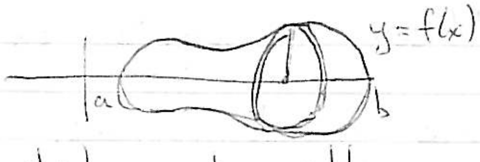
First step into 3-D: solids of revolution, volumes, surface areas

A <u>solid of revolution</u> is what we get when we rotate a plane region around an axis.

<u>E.g.</u>) cylinder, sphere, cone, etc



Volumes via disk method: (rotating around x-axis)



Volume is approximated by a union of disks - so computing volumes of the disks & adding (integrating) gives volume.

$V_{disk} = \underbrace{\pi r^2}_{A} \cdot \underbrace{h}_{dx}$; volume of slice at x is $A(x)\,dx$

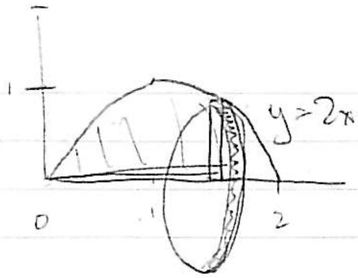
So volume is $\displaystyle\int_a^b A(x)\,dx = \int_a^b \pi f(x)^2\,dx$

in the picture I've drawn

(of course, this formula works for <u>any</u> solid, not just solids of rotation, but computing $A(x)$ can be hard in general.)

<u>Ex</u>: Suppose the region between the x-axis & $y = 2x - x^2$ is rotated around the x-axis. Resulting volume?

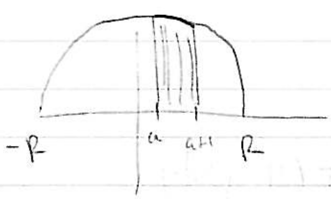
ALWAYS DRAW A PICTURE! (Unless you have really amazing visual intuition :))



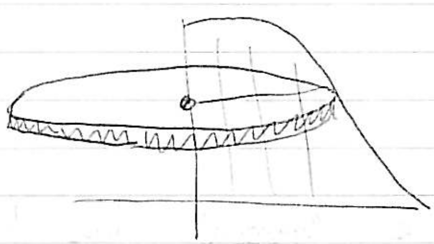
$$V_{slice} = \pi(2x-x^2)^2\,dx$$

$$\Rightarrow \text{total volume} = \int_0^2 \pi(2x-x^2)\,dx$$

Exercise: Suppose $-R \le a$, $a+1 \le R$. What is the volume of the solid formed when rotating the region under $y=\sqrt{R^2-x^2}$ around the x-axis, $a \le x \le a+1$. (A slice of a sphere of thickness 1.)

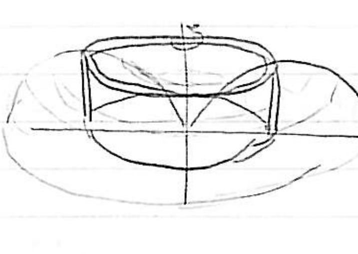


Note: we can also easily adjust this idea to rotate around the y-axis (or any vertical line):



Note: now you're integrating w.r.t. y.

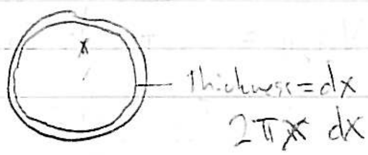
2nd method: Cylindrical shells.



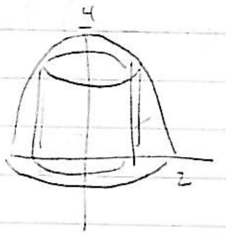
V_{shell} = area · height = $2\pi r\,dx \cdot f(x)$

if rotation centered at y-axis, $r=x$.

(at least in this diagram)



E.g., if region under $y = 4 - x^2$ in 1st quad is revolved around y-axis,

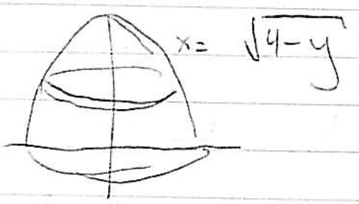


$$V_{shell} = 2\pi x \cdot (4 - x^2)\, dx$$

$$V = \int_0^2 2\pi x\,(4 - x^2)\, dx$$

$$=$$

Double-check w/ disks:



$$V_{disk} = \pi \sqrt{4-y}^{\,2}\, dy = \pi(4-y)\, dy$$

$$\text{Volume} = \int_0^4 \pi(4-y)\, dy$$

$$=$$

(This method is good if our solids are "gappy" like the last one on previous pg -- to do disks w/ such a solid, must take a difference.)

Rule: remember $V_{disk} = \pi r^2 \cdot h$, $h = d(\text{variable of integration})$

$$V_{shell} = 2\pi r h\, dr$$

Draw picture to find the right r, h.

Lecture 4 ended here (2 hrs)

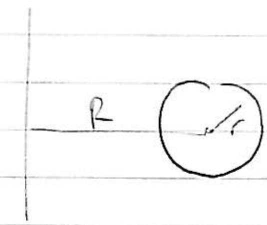
Insert at end of Lecture 4 (after p 12)

Pappus' Theorem:

$V(\text{solid of rotation}) = 2\pi R \cdot A$ where

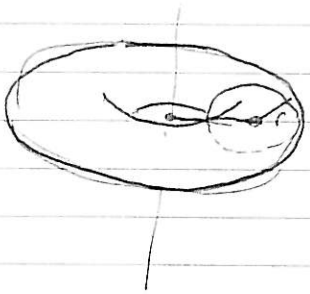
A = area,

R = distance from $\underbrace{\text{center of mass}}_{\text{centroid}}$ to axis of rotation



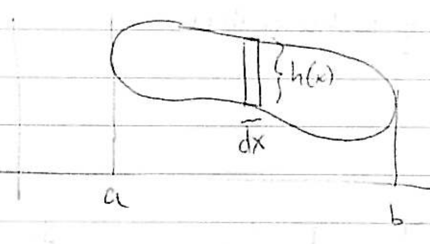
So volume of torus (= donut)

$= 2\pi^2 R r^2$



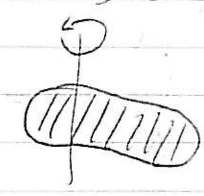
To compute centroid:

$$x\text{-coord centroid} = \int_a^b x \cdot h(x)\,dx \Big/ \int_a^b h(x)\,dx$$



(lower integral = area)

Caution: none of what we've said is valid if we're in this situation:



(Region crosses axis, overlaps when rotated.)