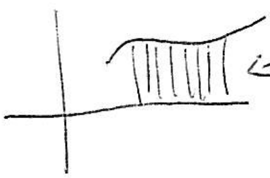
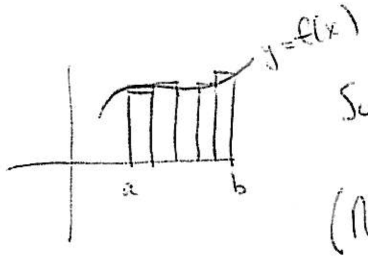
One major tool for computing antiderivatives is left, but detour to <u>Definite Integrals</u> (the 2nd geo. problem).



← area = ?

idea: cut into rectangles, take limit as size of rectangles $\to 0$, # rectangles $\to \infty$.



Sum of areas of rectangles ⟶ area under curve

(Note area = "algebraic area"; if $f(x)<0$, something strange happens.)

<u>E.g.</u>, what is the area under the line $y=2x$ between $x=0$ & $x=a$?

(simple so we can check our answers)

n rectangles; height of ith is $\frac{2ai}{n}$, base is $\frac{a}{n}$

$\Rightarrow$ area is $\frac{2a^2 i}{n^2}$ & total area is

$$\sum_{i=1}^{n} \frac{2a^2 i}{n^2} = \frac{2a^2}{n^2}(1+2+\dots+n) = \frac{2a^2}{n^2}\cdot\frac{n(n+1)}{2} \xrightarrow[n\to\infty]{} a^2.$$

But this method would be ridiculous even for moderately more complicated functions. (Polynomials, okay, maybe exponential functions, but everything else is hopeless.)

So, what to do? Well, we get lucky:

Fundamental Theorem of Calculus: if F is an antiderivative of f then the area under $f(x)$ between $x=a$ & $x=b$ is equal to $F(b)-F(a)$.

We denote this area by $\int_a^b f(x)\,dx$; it is the <u>definite integral</u> of $f(x)$ between a & b. ($\int$ = S for sum). Idea of proof:

$G(t) = \int_a^t f(x)\,dx$ as a function of t. Then its derivative is just (the height of the "marginal rectangle") & since $G(a) = 0$, this gives we want. This also gives us <u>2nd Fund Thm Calc</u>:

$$\frac{d}{dt} \int_a^t f(x)\,dx = f(t)$$

implicit assumptions: f is "well-behaved" - piecewise continuous, bounded, so integrating across big holes or anything.)

<u>Properties of definite integrals:</u>

- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

- $\int_a^a f(x)\,dx = 0$

- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

- $\int_a^b c f(x)\,dx = c \int_a^b f(x)\,dx$

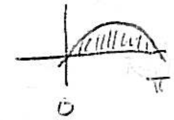
- $\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

- If $f(x) \le g(x)$ for all $x \in [a, b]$, $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

(provided all fns are still well-behaved etc.)
& This is true "in nature"

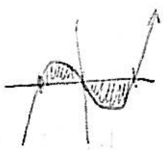
So, let's compute some areas / integrals:

- How much area under one arch of $f(x) = \sin x$?



$$\int_0^{\pi} \sin x \, dx = -\cos x \Big|_{x=0}^{\pi} = 1 - (-1) = 2.$$

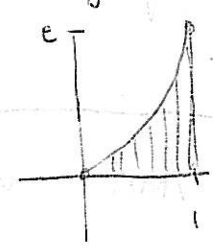
- What is the area bounded between $y = x^3 - x$ & the x-axis?



(Rule: always draw the picture.)

- What is the area between $y = \sqrt{4-x^2}$ & the x-axis?

Note: when making substitutions in definite integrals, we can change bounds of integration & never back-substitute.



Suppose wanted area beneath $y = xe^{x^2}$, $0 \le x \le 1$

One method is $\displaystyle\int x e^{x^2} dx \overset{\substack{u=x^2 \\ du=2x\,dx}}{=} \frac{1}{2}\int e^u \, du = \frac{1}{2}e^u = \frac{1}{2}e^{x^2}$

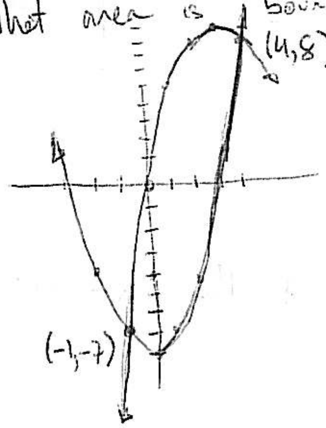
$$\Rightarrow \int_0^1 x e^{x^2} dx = \frac{1}{2}e^{x^2}\Big|_{x=0}^{1} = \frac{1}{2}e - \frac{1}{2}.$$

Other method is $\displaystyle\int_{x=0}^{1} x e^{x^2} dx \underset{\substack{x=0 \Rightarrow u=0 \\ x=1 \Rightarrow u=1}}{\overset{\substack{u=x^2 \\ du=2x\,dx}}{=}} \int_{u=0}^{1} \frac{1}{2}e^u \, du = \frac{1}{2}e^u\Big|_{u=0}^{1} = \frac{1}{2}e - \frac{1}{2}.$

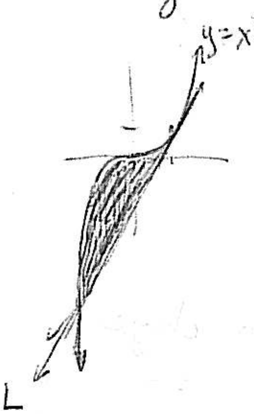
Try something similar for $\displaystyle\int_0^{\pi} \sin t \cos t \, dt$.

Moving on to more complicated problems:

What area is bounded between $y = x^2 - 8$ & $y = -x^2 + 6x$?

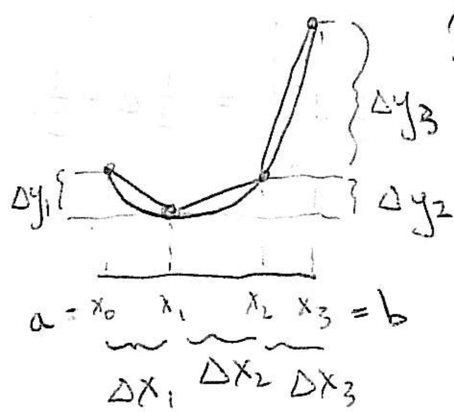


Challenge: What area is bounded between $y = x^3$ & the line tangent to it @ $x=1$?



Another use of integrals: <u>arclength</u>: estimate curve is a polygon,

length of polygon $\approx$ length of curve, & approximation gets better & better as # of edges increases.



Length of polygon $= \sum$ Lengths of chords

$$= \sum_k \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

$$= \sum_k \Delta x_k \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2}.$$

Now in the limit ($n \to \infty$, $\Delta x_k \to 0$ for all k) we have

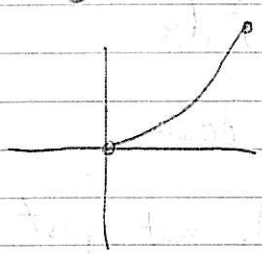
$$\sum_k \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2} \cdot \Delta x_k \longrightarrow \int_a^b \sqrt{1 + f'(x)^2}\, dx .$$

(Easier to remember maybe: $\displaystyle\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$$= \int_a^b \sqrt{\underbrace{(dx)^2 + (dy)^2}_{(ds)^2}} \quad \text{where } s = \text{arclength}$$

Sad fact: arclength is very difficult to compute exactly for many (most) curves — few nice antiderivatives. But there are a few functions for which it's nice:

e.g.) $\quad y = 2x^{3/2} \quad (x > 0) \quad$ for $0 \le x \le 2$



$y' = 3x^{1/2}$

$$S = \int_0^2 \sqrt{1 + (3x^{1/2})^2}\, dx$$

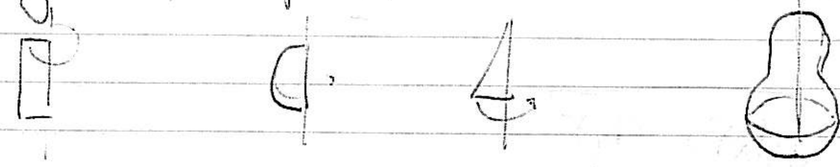
$$= \int_0^2 \sqrt{1 + 9x}\, dx$$

$$=$$

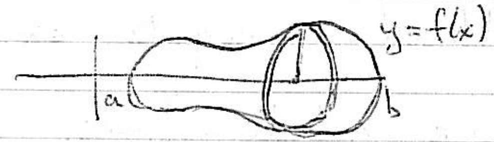
First step into 3-D: solids of revolution, volumes, surface areas

A *solid of revolution* is what we get when we rotate a plane region around an axis.

E.g.) cylinder, sphere, cone, etc



Volumes via disk method: (rotating around x-axis)



Volume is approximated by a union of disks – so computing volumes of the disks & adding (integrating) gives volume.

$V_{disk} = \underbrace{\pi r^2}_{A} \cdot \underbrace{h}_{dx}$; volume of slice at x is $A(x)\,dx$

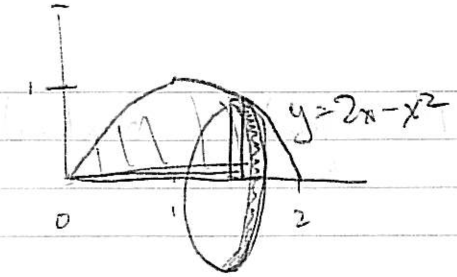
So volume is $\displaystyle\int_a^b A(x)\,dx = \int_a^b \pi f(x)^2\,dx$

in the picture I've drawn

(of course, this formula works for *any* solid, not just solids of rotation, but computing $A(x)$ can be hard in general.)

Ex: Suppose the region between the x-axis & $y = 2x - x^2$ is rotated around the x-axis. Resulting volume?

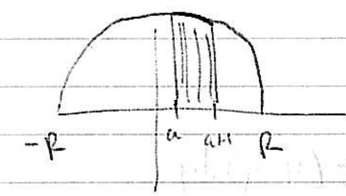
ALWAYS DRAW A PICTURE! (Unless you have really amazing visual intuition ☺)



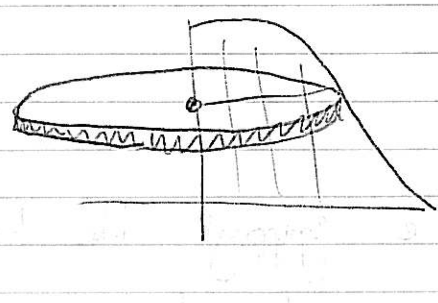
$$V_{slice} = \pi(2x-x^2)^2\,dx$$

$$\Rightarrow \text{total volume} = \int_0^2 \pi(2x-x^2)\,dx$$

__Exercise__: Suppose $-R \le a$, $a+1 \le R$. What is the volume of the solid ~~formed~~ formed when rotating the region under $y = \sqrt{R^2 - x^2}$ around the x-axis, $a \le x \le a+1$. (A slice of a sphere of thickness 1.)

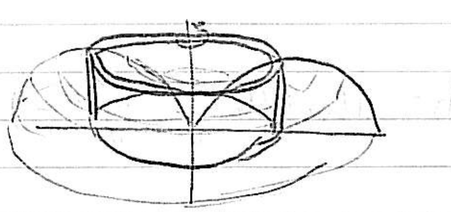


__Note__: we can also easily adjust this idea to rotate around the y-axis (or any vertical line):



Note: now you're integrating w.r.t. y.

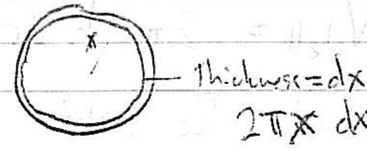
__2nd method__: Cylindrical shells.



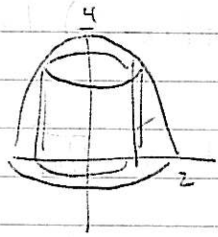
if rotation centered at y-axis, $r = x$.

$$V_{shell} = \text{area} \cdot \text{height} = 2\pi r\,dx \cdot f(x)$$

(at least in this diagram)



E.g., if region under $y = 4 - x^2$ in 1st quad is revolved around y-axis,

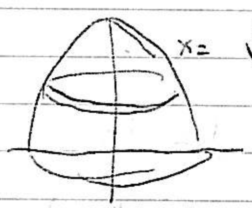


$$V_{shell} = 2\pi x \cdot (4 - x^2)\,dx$$

$$V = \int_0^2 2\pi x\,(4 - x^2)\,dx$$

$$=$$

Double-check w/ disks:



$$V_{disk} = \pi \sqrt{4-y}^{\,2}\,dy = \pi(4-y)\,dy$$

$$Volume = \int_0^4 \pi(4-y)\,dy$$

$$=$$

(This method is good if our solids are "gappy" like the last one on previous pg... to do disks w/ such a solid, must take a difference.)

Rule: remember $V_{disk} = \pi r^2 \cdot h$, $h = d$(variable of integration)

$$V_{shell} = 2\pi r h\,dr$$

Draw picture to find the right r, h.