Vector field associates to each point in some region (plane or 3D) a vector. Our usual fn of 2 or 3 vars (associating to each pt a #) is a scalar field.

Every scalar field $f(x,y)$ gives rise to a vector field $\vec{\nabla} f$

Saw line integral $\int_C \vec{F} \cdot d\vec{R} = \int_C F_1(x,y)\,dx + F_2(x,y)\,dy$

$= \int_a^b F_1(x,y)\,x'(t) + F_2(x,y)\,y'(t)\,dt$

where C is paramet'ized by $\langle x(t), y(t) \rangle$ for $a \le t \le b$.

Saw $\int_C \vec{\nabla} f \cdot d\vec{R} = f(\vec{R}(b)) - f(\vec{R}(a))$

→ path independence

→ conservative field

→ can think of $-f$ as the potential energy function

One consequence: if C is a closed path,

$\oint_C \quad \int_C \vec{\nabla} f \cdot d\vec{R} = 0$ (& if $\int_C \vec{F} \cdot d\vec{R} = 0$ for all closed C, $\vec{F}$ is conservative)

Another: it suffices to figure out f & ignore everything about the path except endpoints.

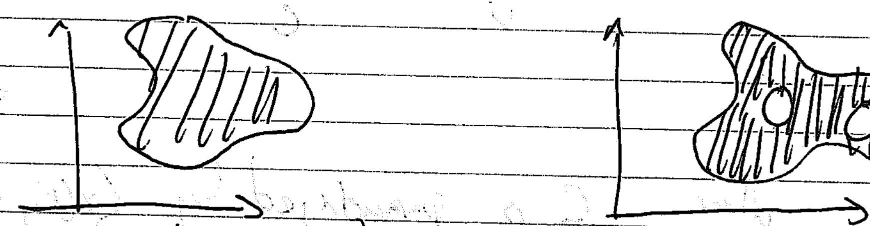
Ex: $\vec{F} = y\cos xy\,\hat{i} + x\cos xy\,\hat{j}$ along the parabola $y = x^2$ going $(0,0)$ to $(1,1)$

$$\vec{F} = \vec{\nabla} f \quad \text{where} \quad f(x,y) = \sin(xy)$$

$$\Rightarrow \int_C \vec{F} \cdot d\vec{R} = f(1,1) - f(0,0) = \sin 1.$$

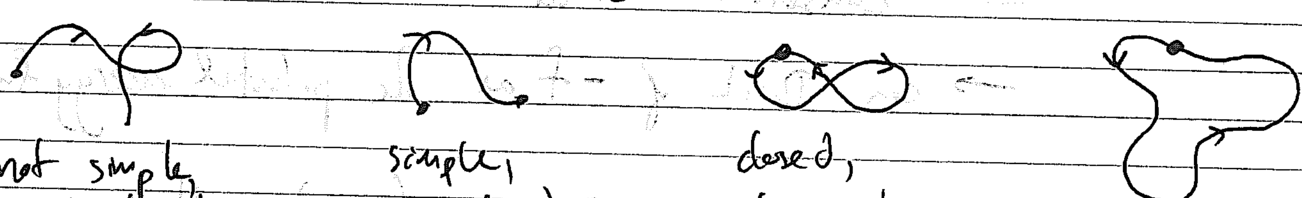
Green's Theorem

Preliminaries:



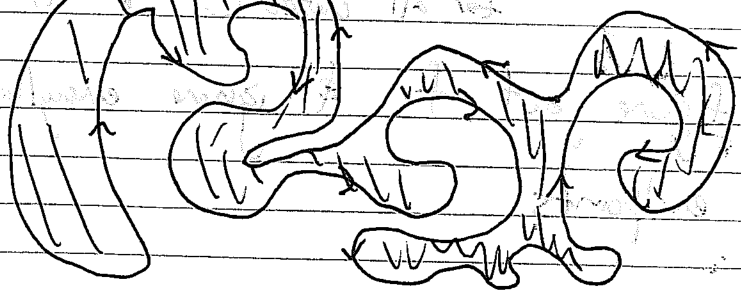
simply connected   Not.

The boundary of a simply connected region is a simple (ie, non-self-intersecting) closed curve



not simple, not closed   simple, not closed   closed, not simple   simple, closed.

Orient boundary: clockwise = negative
counterclockwise = positive. (region always on your left.)



1.2. Green's theorem

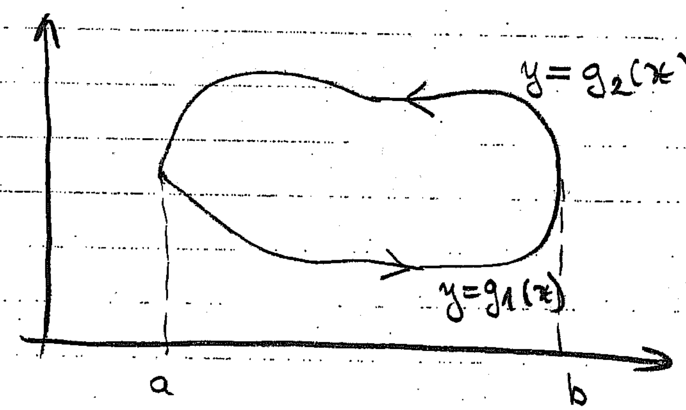
If R is a simply connected region and C is its boundary with the positive orientation, then

$$\boxed{\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$

Obs: Sometimes it is written:

$$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA = \int_C M\,dy - N\,dx$$

Proof



We prove that

$$\int_C P\,dx = -\iint_R \frac{\partial P}{\partial y}\,dA$$

Start with the RHS

$$-\iint_R \frac{\partial P}{\partial y}\,dA = -\int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}\,dy\,dx$$

$$= -\int_a^b P(x,y)\Big|_{y=g_1(x)}^{y=g_2(x)} dx = -\int_a^b \left[P(x,g_2(x)) - P(x,g_1(x))\right] dx$$

We look now at the LHS:

$$\int_C P\,dx = \int_{C_1} P\,dx - \int_{C_2} P\,dx$$

where C_1 is the lower curve $y = g_1(x)$
C_2 is the upper curve $y = g_2(x)$

$$\int_{C_1} P(x,y)\,dx = \int_a^b P(x, g_1(x))\,dx$$

$$\int_{C_2} P(x,y)\,dx = \int_a^b P(x, g_2(x))\,dx$$

$$\Rightarrow \text{LHS} = \int_a^b [P(x,g_1(x)) - P(x,g_2(x))]\,dx = \text{RHS}.$$

In a similar way one proves that

$$\int_C Q\,dy = \iint_R \frac{\partial Q}{\partial x}\,dA$$

Adding up the two equalities one obtains Green's theorem.

Remark: One easy way to remember the RHS of Green's theorem is: the integrand is the determinant

$$\begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ P & Q \end{vmatrix}$$

If we define the vector field $\vec{F}(x,y) = (P(x,y), Q(x,y))$ then

$$\text{curl}\,\vec{F} = \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ P & Q \end{vmatrix}$$