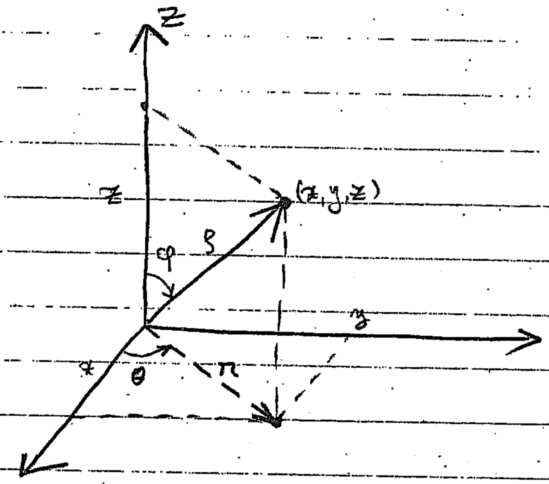
18.089

Lecture 17 (06/27/03)

1. Spherical coordinates



For $0 \leq \phi \leq \pi$, $0 \leq \theta \leq 2\pi$, $0 \leq \rho$

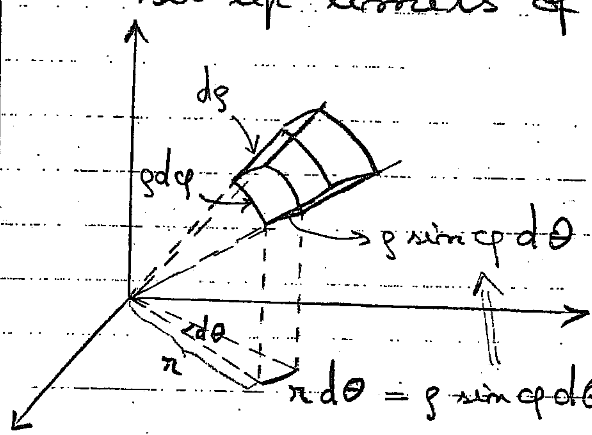
$$\begin{cases} x = \rho \sin\varphi \cos\theta & (= r\cos\theta) \\ y = \rho \sin\varphi \sin\theta & (= r\sin\theta) \\ z = \rho\cos\varphi \end{cases}$$

(Connection with cylindrical coordinates:

$$\begin{cases} r = \rho \sin\varphi \\ z = \rho\cos\varphi \\ \theta = \theta \end{cases}$$
)

To calculate $\iiint_D f \, dV$ we have to:

- express dV in terms of $d\rho$, $d\varphi$, $d\theta$
- set up limits of integration



$$dV = d\rho \cdot (\rho \, d\varphi) \cdot (\rho \sin\varphi \, d\theta)$$

$$\boxed{dV = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta}$$

We can derive the same formula starting from the formula of dV in cylindrical coordinates:

$$dV = dz \cdot r\,dr\,d\theta$$

We have: $\begin{cases} r = \rho \sin\varphi \\ z = \rho\cos\varphi \end{cases} \Rightarrow dz\,dr = \rho\,d\rho\,d\varphi$ (just like for polar coordinates)

$$dV = r\,(dz\,dr)\,d\theta = $$
$$= (\rho\sin\varphi)\cdot(\rho\,d\rho\,d\varphi)\,d\theta$$
$$= \rho^2 \sin\varphi\,d\rho\,d\varphi\,d\theta$$

Setting up limits of integration

$$\boxed{\iiint_D f\,dV = \int_{\theta_1}^{\theta_2}\int_{\varphi_1(\theta)}^{\varphi_2(\theta)}\int_{\rho_1(\theta,\varphi)}^{\rho_2(\theta,\varphi)} f(\rho,\varphi,\theta)\,d\rho\,d\varphi\,d\theta}$$

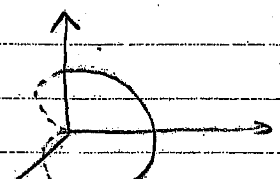
As before:

- the first set of limits are constants: θ_1, θ_2
- the second set of limits are functions of one variable: $\varphi_1(\theta), \varphi_2(\theta)$
- the third set of limits are functions of two variables: $\rho_1(\theta,\varphi), \rho_2(\theta,\varphi)$

- integration is done from inside out.

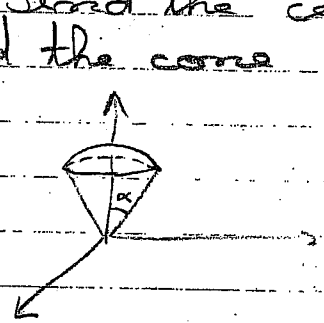
Examples

a) One takes the cardioid $s = 1+\cos\psi$, $-\frac{\pi}{2} \le \psi \le \frac{\pi}{2}$ and rotates it around the z-axis. The interior of this surface is D.

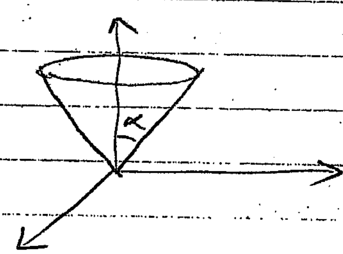


$$\iiint_D f\,dV = ?$$

b) Find the centroid bounded by the sphere $\rho = a$ and the cone $\varphi = \alpha$ (Simmons 741)



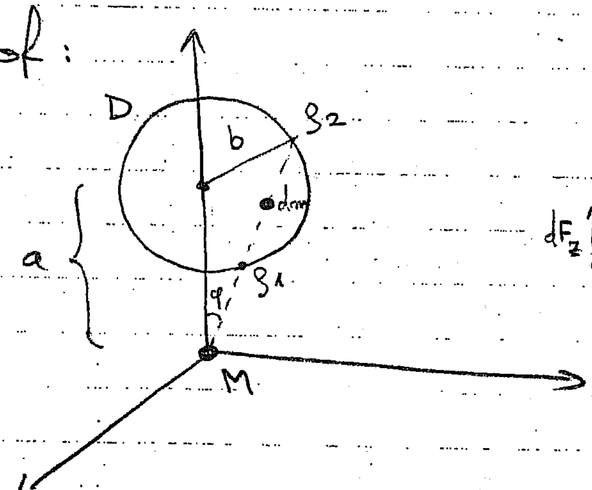
c) Write limits of integration for:
- sphere of radius a
- upper hemisphere of radius a
- the cone:

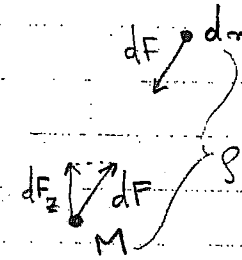


2. Gravitational attraction

Newton's theorem: a solid sphere of uniform density attracts as if all its mass were concentrated at its center.

Proof:





$$dF = \frac{GM\,dm}{\rho^2}$$

$$dF_z = dF \cdot \cos\varphi$$

Total gravitational force $\iiint_D dF_z = \iiint_D \frac{GM\delta}{\rho^2}\cos\varphi\, dV$

$$= \int_0^{2\pi}\int_0^{\phi_0}\int_{\rho_1}^{\rho_2} \frac{GM\delta}{\cancel{\rho^2}}\cos\varphi \cdot \cancel{\rho^2}\sin\varphi\, d\rho\, d\varphi\, d\theta$$

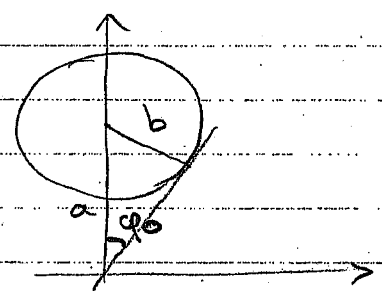
We have to find ρ_1, ρ_2 and φ_0.

Law of cosines: $\rho^2 + a^2 - 2\rho a \cos\varphi = b^2$

$$\rho = a\cos\varphi \pm \sqrt{a^2\cos^2\varphi - (a^2 - b^2)}$$

$$\rho_1 = a\cos\varphi - \sqrt{b^2 - a^2\sin^2\varphi}$$

$$\rho_2 = a\cos\varphi + \sqrt{b^2 - a^2\sin^2\varphi}$$



$$\sin\varphi_0 = \frac{b}{a}$$

$$F = GM\delta \int_0^{2\pi}\int_0^{\varphi_0} \cos\varphi \sin\varphi \, \rho\Big|_{\rho_1}^{\rho_2} d\varphi \, d\theta =$$

$$= GM\delta \int_0^{2\pi} \Big(\underbrace{\int_0^{\varphi_0} 2\sqrt{b^2 - a^2\sin^2\varphi} \, \cos\varphi \sin\varphi \, d\varphi}_{\text{doesn't depend on } \theta!} \Big) d\theta$$

$$= GM\delta \cdot 2\pi \int_0^{\varphi_0} 2\sqrt{b^2 - a^2\sin^2\varphi} \, \cos\varphi \sin\varphi \, d\varphi$$

$$= GM\delta \cdot 2\pi \cdot a \int_0^{\varphi_0} \sqrt{\left(\frac{b}{a}\right)^2 - \sin^2\varphi} \; 2\cos\varphi \sin\varphi \, d\varphi$$

$$u = \left(\frac{b}{a}\right)^2 - \sin^2\varphi \,, \quad du = -2\cos\varphi \sin\varphi \, d\varphi$$

$$\varphi = 0 \Rightarrow u = \left(\frac{b}{a}\right)^2 \,, \quad \varphi = \varphi_0 \Rightarrow u = 0$$

$$= -GM\delta \cdot 2\pi a \int_{\left(\frac{b}{a}\right)^2}^{0} u^{\frac{1}{2}} \, du = 2\pi GM\delta a \frac{\left[\left(\frac{b}{a}\right)^2\right]^{\frac{3}{2}}}{\frac{3}{2}}$$

$$= 2\pi G M \delta a \cdot \frac{2}{3} \cdot \frac{b^3}{a^3} = GM \cdot \underbrace{\frac{4\pi b^3}{3} \cdot \delta}_{M_{sphere}} \cdot \frac{1}{a^2}$$

$$= \frac{G M M_{sphere}}{a^2}$$ which is what we wanted.

3. Change of variables.

We've seen so far how to change the rectangular coordinates into
- polar coordinates (2d)
- cylindrical and spherical coordinates (3d)

But what if for some problems there are other coordinates which arise more naturally. Is there a theory concerning, say parabolic or catenoidal coordinates?

Let's recall the one-variable change of coordinates:

$$\int_a^b f(x)\,dx = \int_c^d f(\varphi(u))\,\varphi'(u)\,du$$

But this is just a formula. In real life we set this up like this:

$$\int_a^b f(x)\,dx = ?$$

$x = x(u)$ $\quad dx = x'(u)\,du$

$x = a \Rightarrow u = c$

$x = b \Rightarrow u = d$

$$\int_a^b f(x)\,dx = \int_c^d f(x(u))\,x'(u)\,du$$

When we have two variables we'll do similarly:

$$\iint_R f(x,y)\,dx\,dy$$

$$x = x(u,v) \Rightarrow dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv$$
$$y = y(u,v) \Rightarrow dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv$$

$$\begin{aligned} dx\,dy &= (x_u\,du + x_v\,dv)(y_u\,du + y_v\,dv) \\ &= \cancel{x_u y_u\,du\,du} + x_u y_v\,du\,dv + x_v y_u\,dv\,du + \\ &\quad + \cancel{x_v y_v\,dv\,dv} \\ &= (x_u y_v - x_v y_u)\,du\,dv = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} du\,dv \end{aligned}$$

The determinant $\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$ is often denoted by $\frac{\partial(x,y)}{\partial(u,v)}$ and is called the Jacobian of the change of variables. So we have the following form of the chain rule:

$$dx\,dy = \frac{\partial(x,y)}{\partial(u,v)}\,du\,dv$$

The integral becomes

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(x(u,v), y(u,v))\frac{\partial(x,y)}{\partial(u,v)}\,du\,dv$$

The region of integration S is the set of all pairs (u,v) such that $(x(u,v), y(u,v)) \in R$.

Examples:

a) Use change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral $\iint_R y\,dA$, where R is the region bounded by the x-axis and the parabolas

b) Evaluate the integral $\iint_R e^{\frac{x+y}{x-y}}\, dA$ where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,-2)$, $(0,-1)$.

The change of variables for triple integrals is similar:

$$\iiint_R f(x,y,z)\, dV = \iiint_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \cdot \frac{\partial(x,y,z)}{\partial(u,v,w)}\, du\, dv\, dw$$

where

$$\frac{\partial(x,y,v)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Homework

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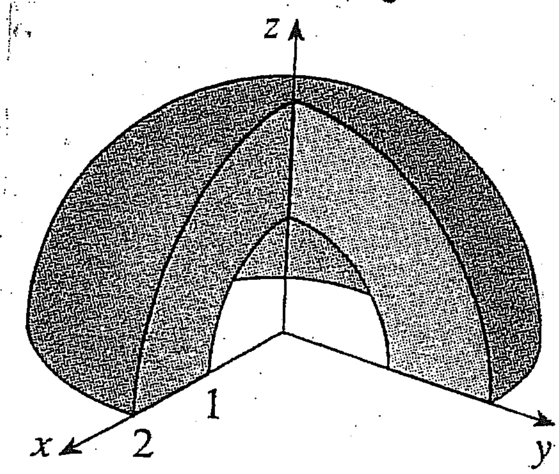
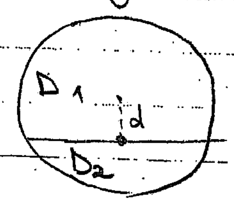
B Set up the integral $\iiint_R f\, dV$ for R as below:



744/18 You have to calculate two integrals:



$F_1 = \iiint_{D_1} (\ldots)\, dV$, $F_2 = \iiint_{D_2} (\ldots)\, dV$

$F = F_1 - F_2 \approx d$

c. $\iint_R \frac{x+2y}{\cos(x-y)}\, dA$, R the parallelogram bounded by the lines $y=x$, $y=x-1$, $x+2y=0$, $x+2y=2$.

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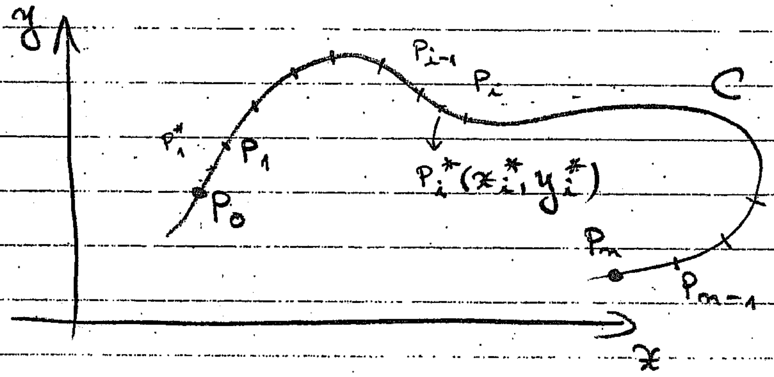
Lecture 18 (06/30/03)

1. Line integrals

1.1. Line integrals with respect to arclength

Natural generalization of the one-variable integral.



If we start with a curve $x = x(t)$, $y = y(t)$, $t \in [a,b]$ we want to integrate a function $f(x,y)$ along the curve, we pick points $P_0, P_1, \dots, P_n$ and then sampling points $P_1^*, P_2^*, \dots, P_n^*$ and define the Riemann sum

$$\sum f(x_i^*, y_i^*) \Delta s_i$$

where Δs_i = length of arc $P_{i-1} P_i$.

$$\int_C f(x,y)\, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*) \Delta s_i$$

We will not use this definition for computation. We remember that:

$$ds^2 = dx^2 + dy^2 = \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] dt^2 \Rightarrow$$

$$\Rightarrow ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

The formula for computing line integrals with respect to arclength:

$$\boxed{\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt}$$

Examples:

a) $\int_C (2+x^2y)\,ds$, C is the left half of the unit circle

b) $\int_C 2x\,ds$, C is the arc C_1 of the parabola $y=x^2$ from $(0,0)$ to $(1,1)$ followed by the vertical line segment C_2 from $(1,1)$ to $(1,2)$

1.2. The line integral of a differential 1-form.

$$\int_C P(x,y)\,dx + Q(x,y)\,dy$$

To compute this we just replace dx by $x'(t)\,dt$ and dy by $y'(t)\,dt$

$$\boxed{\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_a^b \left[P(x(t),y(t))\,x'(t) + Q(x(t),y(t))\,y'(t)\right]dt}$$

Examples

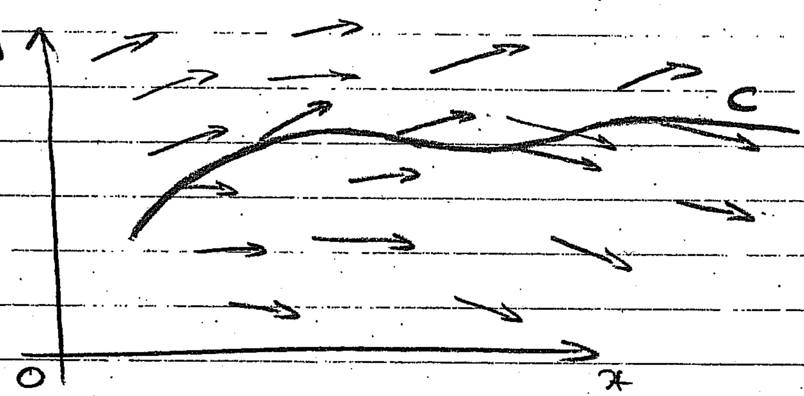
c) $\int_C y^2\,dx + x\,dy$ where C = the line segment with vertices $(-5,-3)$, $(0,2)$

d) the same integral for C = the arc of parabola $x=4-y^2$ from $(-5,-3)$ to $(0,2)$.

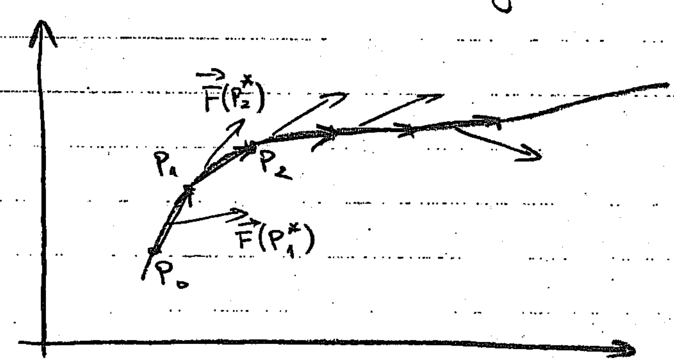
e) $\int_C y\,dx + z\,dy + x\,dz$, C - the line segment from $(2,0,0)$ to $(3,4,5)$.

1.3. Line integrals of vector fields

The guiding example is of a particle moving in a force field.



We want to calculate the work done by the force field in moving the particle. Again we divide the curve into small arcs and consider that these arcs are segments



$$\sum \vec{F}(P_i^*) \cdot \overrightarrow{P_{i-1}P_i}$$

Again

$$\int_C \vec{F}(x,y)\, d\vec{P} = \lim_{n\to\infty} \sum_{i=1}^{n} \vec{F}(\vec{P}_i^*) \cdot \overrightarrow{P_{i-1}P_i}$$

but this is not too useful for calculations.

Instead: $\vec{P} = (x,y) \Rightarrow d\vec{P} = (dx, dy)$, and
$x = x(t)$, $y = y(t) \Rightarrow dx = x'(t)\,dt$, $dy = y'(t)\,dt$
$\Rightarrow d\vec{P} = (x'(t)\,dt; y'(t)\,dt)$

$$\vec{F} = (\vec{F_1}, \vec{F_2}) \Rightarrow \vec{F} \cdot d\vec{P} = (F_1, F_2)(x'\,dt, y'\,dt) \Rightarrow$$

$$\Rightarrow \int_C \vec{F}\, d\vec{P} = \int_a^b [F_1(x(t), y(t))\, x'(t) + F_2(x(t), y(t))\, y'(t)]\, dt.$$

You don't really need to remember this formula, just remember how we derived it.

Examples:

f) Find the work done by the force field $\vec{F}(x,y) = (x^2, -xy)$ in moving a particle along the quarter-circle $\vec{P}(t) = (\cos t, \sin t)$, $0 \le t \le \frac{\pi}{2}$.

g) $\int_C \vec{F} \cdot d\vec{P}$, $\vec{F}(x,y,z) = (xy, yz, zx)$
$\vec{P}(t) = (t, t^2, t^3)$, $0 \le t \le 1$.

2. Fundamental theorem of line integrals.

This is the analogue of the fundamental theorem of calculus in the case of line integrals.

$$\boxed{\int_C \vec{\nabla} f\, d\vec{P} = f(\vec{P}(b)) - f(\vec{P}(a)) = f\Big|_{\vec{P}(a)}^{\vec{P}(b)}}$$

Proof:

$$\int_C \vec{\nabla} f\, d\vec{P} = \int_C \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot (dx, dy) =$$

$$= \int_C \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy$$

$$= \int_a^b \left(\frac{\partial f}{\partial x} \frac{dx}{dt}\, dt + \frac{\partial f}{\partial y} \frac{dy}{dt}\, dt\right) = \int_a^b \frac{d}{dt} f(\vec{P}(t))\, dt$$

$$= f(\vec{P}(b)) - f(\vec{P}(a))$$

Remark: This proof shows us also that:

$$\int_c df = f(\vec{P}(b)) - f(\vec{P}(a)).$$

Remark: This theorem tells us two things:
- the integral of a gradient field over a curve depends only on the endpoints of the curve. This property is known as the path independence property of a gradient field.
- the integral over a closed curve of a gradient field is always 0.

These two conditions are equivalent. Any field satisfying one of the two properties is called a conservative field. The fundamental theorem for line integrals says that any gradient field is a conservative field. The converse is also true, the proof can be found in Simmons p.760.

Examples

i) (Simmons 763/2) $\vec{F} = (x(y-1), x)$ is not conservative

j) (Simmons 763/9) Show that

$$\int_{(-2,1)}^{(1,4)} 2xy\,dx + x^2\,dy$$

is independent of path and evaluate the integral first by using the fundamental theorem of line integrals, then by choosing a convenient path and integrating along it.

Homework 9

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D: $\int_c xy^4\,ds$, c the right half of circle $x^2+y^2=16$