Basic properties of exponential functions & logarithms

$a^{m+n} = a^m \cdot a^n$ $\qquad$ $\log_a(m \cdot n) = \log_a(m) + \log_a(n)$

$a^{m \cdot n} = (a^m)^n = (a^n)^m$ $\qquad$ $\log_a(m^n) = n \cdot \log_a(m)$

$\log_a 1 =$ $\qquad$ $\log_a a =$

Special constant $e \approx 2.718281828459 0 \ldots$ (irrational - no pattern in the digits).
$= \lim_{n\to\infty} (1+1/n)^n$.

We call $\exp(x) = e^x$ <u>the</u> exponential function &
$\ln(x) = \log_e x$ <u>the</u> logarithm (or commonly the <u>natural logarithm</u>)

These functions are very special, for the following reasons:

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(\ln x) = \frac{1}{x}$$

(question: do we know the function whose derivative is x^r?)

<u>Remark</u>: It's enough to work only with exp & ln, because we can write all exponential & logarithmic functions in terms of these two:

$$a = e^{\ln a} \Rightarrow a^x = e^{x \cdot \ln a}, \quad \text{\& similarly} \quad \log_a x = \frac{\ln x}{\ln a}.$$

<u>Examples</u>: $\frac{d}{dx}(2^x) = ?$ $\qquad$ $\frac{d}{dx}\left(e^{x+x^2}\right) = ?$

$\frac{d}{dx}\left(\ln((x-1)^2)\right) = ?$ $\qquad$ $\frac{d}{dx}\left(\ln(\ln(x))\right) = ?$

Trigonometric functions

Remark: For calculus, all angles are <u>always</u> in radians, NOT degrees.

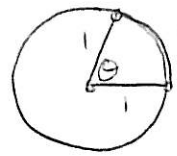
$360^\circ = 2\pi$ radians

$60^\circ = \pi/3$ radians

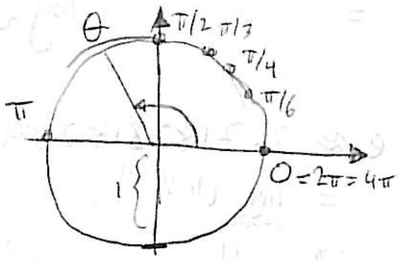
$30^\circ = \pi/6$ radians

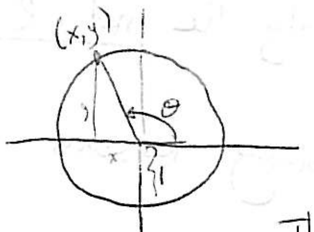
$90^\circ = \pi/2$

$45^\circ = \pi/4$



arclength $= \theta$
(when θ measured in radians).

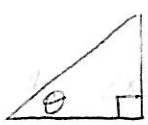




$$\left.\begin{aligned} x &= \cos\theta \\ y &= \sin\theta \end{aligned}\right\}$$ <u>definitions</u>.

Then can define the other trig functions

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$



In a right triangle (when $0 < \theta < \pi/2$), these are all the possible ratios of pairs of sides — SOH CAH TOA

Important trig identities

$\sin^2\theta + \cos^2\theta =$

$\sin(-\theta) =$

$\cos(-\theta) =$

$\sin(2\theta) =$

$\cos(2\theta) =$

notation: we write $\sin^2\theta$ for $(\sin\theta)^2$

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\cos\theta$					
$\sin\theta$					
$\tan\theta$					

<u>Graph these</u>

Trig & calculus.

Key limit: $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$. $\lim_{\theta \to 0} \frac{\tan\theta}{\theta} =$

$\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} =$ $\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta^2} =$.

$\frac{d}{dx}(\sin x)\Big|_{x=0} = 1.$ In fact, $\frac{d}{dx}(\sin x) = \cos x$

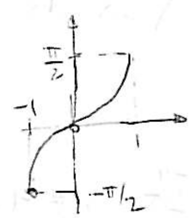
& $\frac{d}{dx}(\cos x) = -\sin x.$ $\frac{d}{dx}(\tan x) =$

$\frac{d}{dx}(\ln(\cos x)) =$ $\frac{d}{dx}(x^3 \sin 2x) =$

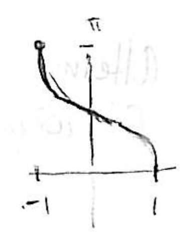
What is the tangent line to $y = \sin x$ @ $x = 0$?

<u>Inverse trig functions</u>: Note that sine & cosine have range $[-1,1]$, tangent has range $(-\infty, \infty) = \mathbb{R}$.

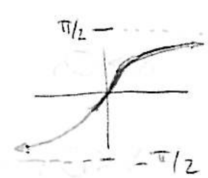
$\sin^{-1} = \arcsin : [-1,1] \longrightarrow [-\pi/2, \pi/2]$



$\cos^{-1} = \arccos : [-1,1] \longrightarrow [0, \pi]$



$\tan^{-1} = \arctan : \mathbb{R} \longrightarrow (-\pi/2, \pi/2)$



$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}(\arccos x) = \frac{-1}{\sqrt{1-x^2}}$ $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$

Questions: $\arcsin(\cos(x)) =$ (assuming $0 \le x \le \pi/2$)

Antiderivatives

If $\frac{d}{dx} f(x) = g(x)$, we say that f is <u>an antiderivative</u> of g (on some interval where is true).

<u>Examples</u>: • If $f'(x) = 0$, what can $f(x)$ be?

<u>Ans</u>: The only such functions are ______ (<u>Mention Mean Value Thm</u>)

• If $f'(x) = 2x$, $f(x) = x^2$. Other solutions? Could have $f(x) = x^2 + c$. Are these only solutions?

Suppose $g'(x) = 2x$

$$- \frac{d}{dx}(x^2) = 2x$$

$$\frac{d}{dx}(g(x) - x^2) = 0 \Rightarrow g(x) - x^2 = c \text{ for some } c,$$

$g(x) = x^2 + c.$

<u>Notation</u>: $\int 2x \, dx = x^2 + c$ ← "the constant of integration" "arbitrary constant"

Exercises! If $n \neq -1$, $\int x^n \, dx =$

For $x > 0$, $\int \frac{1}{x} \, dx =$

For $x < 0$, $\int \frac{1}{x} \, dx =$

$\int e^x \, dx =$

$\int \sin x \, dx =$

$\int \cos x \, dx =$

$\int \sec^2 x \, dx =$

Important note: elementary functions are those that we get from polynomials, logs, exponentials, trig functions by combining them in various ways, e.g., $\sin x \cdot e^{x^2+1} - \ln x$. Our rules tell us how to differentiate any elementary function. Antidifferentiation is much harder & not always possible in terms of elementary functions, e.g., even e^{x^2} doesn't have an elementary anti-derivative.

Antidifferentiation by substitution:

Consider the eqn $\int 3u^2\,du = u^3$. If $u = u(x)$ is a function of x

then $\underbrace{du}_{\text{the differential of } u} = \frac{du}{dx} \cdot dx = u'(x)\,dx$; so we have $\int 3u(x)^2 u'(x)\,dx = u(x)^3$,

& the differential of x

e.g. $\int 3\sin^2 x \cos x\,dx = \sin^3 x$.

Some other examples: • $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx \overset{\substack{u=\cos x\\ du=-\sin x\,dx}}{=} \int \frac{-du}{u}$

(do examples on next page ~~first~~)

$= -\ln|u| + C = -\ln|\cos x| + C$.

• $\int \sin 2x\,dx \overset{\substack{u=2x\\ du=2dx}}{=} \int \sin u \cdot \frac{du}{2} = \frac{1}{2}\int \sin u\,du = -\frac{1}{2}\cos u + C = -\frac{1}{2}\cos 2x + C$

• $\int \frac{1}{1+t^2}\,dt \overset{\substack{t=\tan u\\ dt=\sec^2 u\,du}}{=}$

• $\int e^x \cdot e^{e^x}\,dx =$

• $\int \frac{1}{\sqrt{1-w^2}}\,dw =$

Many (not all) antiderivatives can be computed by a good choice of substitution. (More examples on HW.)

$$\int v^2 e^{v^3}\, dv =$$

$$\int \frac{2v}{v^2+1}\, dv =$$

Now that we've seen some integration using trig substitution, we will show how to use it to get a big family of integrals.

Completing the square:

$$\underbrace{x^2-2x-3}_{} = (x^2-2x+1) - 4 = \underbrace{(x-1)^2-4}_{}$$

$$\underbrace{x^2+4x+5}_{} = (x^2+4x+4) + 1 = \underbrace{(x+2)^2+1}_{}$$

So what? Well, consider

$$\int \frac{dx}{x^2+4x+5} \overset{\substack{u=x+2\\ du=dx}}{=} \int \frac{du}{u^2+1}$$

trig substitution

$$= \arctan(u)+c$$

$$= \arctan(x+2)+c.$$

Similarly,

$$\int \frac{x+2}{\sqrt{3+2x-x^2}}\, dx \overset{\substack{2u=x-1\\ 2du=dx}}{=} \int \frac{2u+3}{\sqrt{4-4u^2}}\cdot 2du$$

$$= \int \frac{2u}{\sqrt{1-u^2}}\, du + 3\int \frac{1}{\sqrt{1-u^2}}\, du$$

$$= -2\sqrt{1-u^2} + 3\arcsin u + c$$

$$= -\sqrt{3+2x-x^2} + 3\arcsin\left(\frac{x-1}{2}\right) + c$$

This suggests an approach to antidifferentiating anything with a denominator like ax^2+bx+c or $\sqrt{ax^2+bx+c}$: complete the square & replace a general quadratic with u^2+k^2, u^2-k^2 or k^2-u^2 for some constant u. Then make respectively a tan, sec or sin substitution for u & integrate. (No general method can exist for square roots of higher-degree polynomials – §10.5 & A.9)

restate trig identities

Now we tackle antiderivatives of rational functions in general: partial fractions.

Question: $\int \frac{2}{x^2-1}\,dx =$ $\quad x=\sec u \quad x^2-1=\tan^2 u$

$dx = \frac{\sin u}{\cos^2 u}\,du$

$= \int \frac{1}{\sin u}\,du$. Haven't learned this one yet, but there's a "trick" to it: $\frac{1}{\sin u} = \csc u = \frac{(\csc u+\cot u)\csc u}{\csc u+\cot u}$, &

since $\frac{d}{du}(\csc u) =$ & $\frac{d}{du}(\cot u) =$ we can rewrite

$$\int \frac{2}{x^2-1}\,dx = 2\int \frac{-d(\csc u+\cot u)}{\csc u+\cot u} = -2\ln(\csc u+\cot u)+C =$$

$$= -\ln\left((\csc u+\cot u)^2\right)+C = -\ln\left(\csc^2 u+2\csc u\cot u+\cot^2 u\right)+C$$

$= -\ln\left(\frac{1+2\cos u+\cos^2 u}{\sin^2 u}\right)+C$. Now $\cos u = \frac{1}{x} \Rightarrow \sin^2 u = 1-\frac{1}{x^2} = \frac{x^2-1}{x^2}$,

so $= -\ln\left(\frac{\left(\frac{1+2x+x^2}{x^2}\right)}{\left(\frac{x^2-1}{x^2}\right)}\right)+C = \ln\left(\frac{x-1}{x+1}\right)+C$. Whew! Easier

method: $\frac{2}{x^2-1} = \frac{1}{x-1}-\frac{1}{x+1} \Rightarrow \int\frac{2}{x^2-1}\,dx = \int \frac{1}{x-1}-\frac{1}{x+1}\,dx$

$= \ln(x-1)-\ln(x+1)+C$ ∎

In general, ANY rational function $\frac{P(x)}{Q(x)}$ can be written as

$$\frac{P(x)}{Q(x)} = \underline{\text{some polynomial}} + \frac{R(x)}{Q(x)}, \quad \deg(R) < \deg(Q).$$ Q can be written

as $$Q(x) = a(x-r_1)^{a_1} \cdot (x-r_2)^{a_2} \cdot \ldots \cdot (x-r_k)^{a_k} \cdot \underbrace{(x^2+b_1x+c_1)^{d_1} \cdot \ldots \cdot (x^2+b_jx+c_j)^{d_j}}_{\text{do not factor over } \mathbb{R}.}$$

Then $$\frac{R(x)}{Q(x)} = \frac{A}{x-r_1} + \frac{B}{(x-r_1)^2} + \cdots + \frac{C}{(x-r_1)^{a_1}} + \frac{D}{(x-r_2)} + \frac{E}{(x-r_2)^2} + \cdots + \frac{F}{(x-r_2)^{a_2}} + \cdots$$

$$+ \frac{Gx+H}{x^2+b_1x+c_1} + \frac{Ix+J}{(x^2+b_1x+c_1)^2} + \cdots$$ for some constants $A, B, \ldots$

& then we can antidifferentiate all the pieces.

$$\frac{4x^2}{x^4-16} =$$

$$\frac{1}{(x^2+2x+2)(x^2-2x+1)} =$$

$$\frac{x^3+x^2}{x^4-2x^3+2x-1} =$$

So it follows that we can antidifferentiate all these functions. (Choose one.)

Easy way to self-check: # undetermined coefficients = degree of denominator.