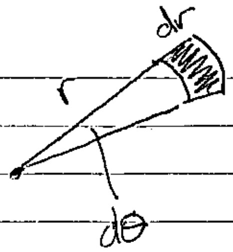
Note relation to previously-used element of area: $\int r\,dr = \frac{1}{2}r^2$!

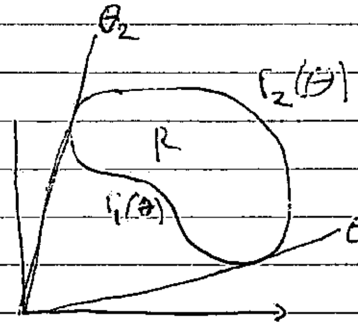


& it has area $dA = r\,dr\,d\theta$.

$(= r\,d\theta\,dr)$ but this is much less useful.

Thus $\iint_R f\,dA = \iint_R f\,dx\,dy$

$= \iint_R f\, r\,dr\,d\theta.$

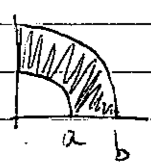


To integrate f over R,

$$\iint_R f\,dA = \int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)} f \cdot r\,dr\,d\theta$$

don't forget this!

Examples: $\iint_R x\,dA$

where $R =$ 

- Find the volume of a sphere using polar integration.

Neat trick: know $\int e^{-x^2}dx$ not elementary. But we will compute $\int_0^\infty e^{-x^2}dx$ as follows:

$$I = \int_0^\infty e^{-x^2}dx = \int_0^\infty e^{-y^2}dy$$

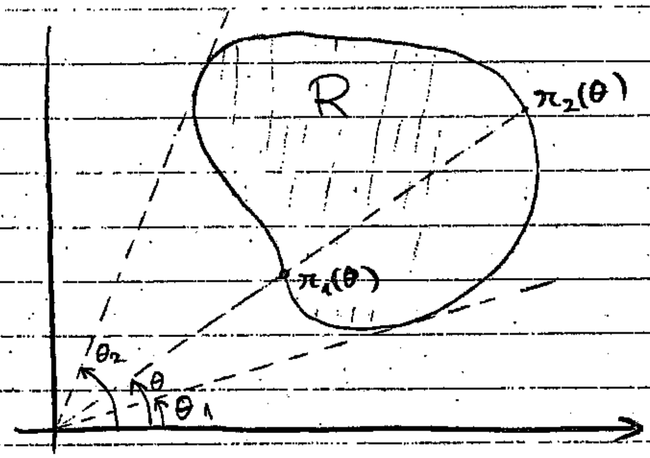
$$\Rightarrow I^2 = \int_0^\infty e^{-x^2}dx \cdot \int_0^\infty e^{-y^2}dy = \int_0^\infty\int_0^\infty e^{-x^2-y^2}dx\,dy$$

18.089

Lecture 16 (06/26/03)

1. More on integrals using polar coordinates

Setting up the limits of integration.



First determine θ_1 and θ_2.

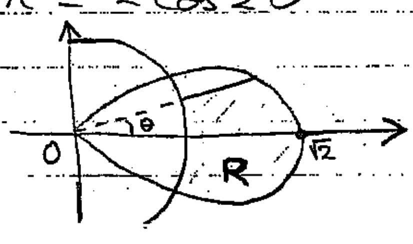
Then, for each θ determine the intersection of the radial line of angle θ with the region R. That means find where this line enters the region, $r_1(\theta)$, and where it leaves it, $r_2(\theta)$.

The integral is:

$$\boxed{\iint_R f\, dA = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta)\, r\, dr\, d\theta}$$

Attention: Do not forget the r before the dr!

Example $\iint_R f\, dA$ where R is the region outside circle of radius 1 and inside the lemniscate $r^2 = 2\cos 2\theta$



Where do the lemniscate and the circle intersect?

$r^2 = 2\cos 2\theta$, $r^2 = 1 \Rightarrow 2\cos 2\theta = 1 \Rightarrow$

$\Rightarrow 2\theta = \pm\frac{\pi}{3} \Rightarrow \theta = \pm\frac{\pi}{6}$

For any $\theta \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$: $r_1(\theta) = 1$
$r_2(\theta) = \sqrt{2\cos 2\theta}$

$$\Rightarrow \iint_R f\, dA = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \int_1^{\sqrt{2\cos 2\theta}} f(r,\theta)\, r\, dr\, d\theta$$

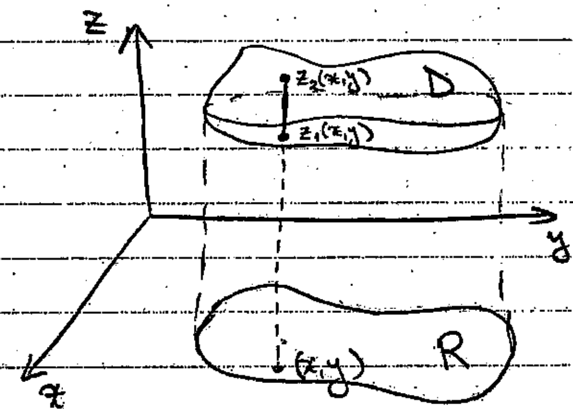
2. Triple integrals

Object of study: $\iiint_D f(x,y,z)\,dV$ (V for volume)

They are very similar to double integrals. For double integrals we looked at 3 things:
- definition
- Fubini's theorem
- polar coordinates

We won't discussed about the definition again. It's completely similar, only that we replace the rectangles with boxes.

3. Fubini's theorem for triple integrals.

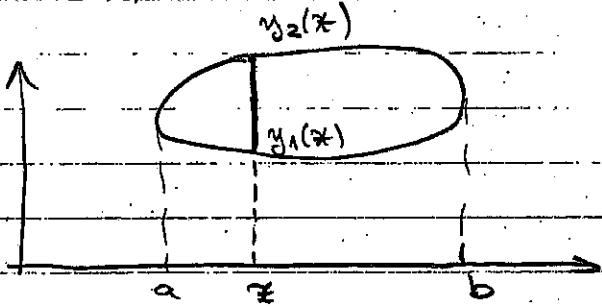


First take R the projection (shadow) of D on the xy-plane.

For each $(x,y) \in R$ find $z_1(x,y)$ and $z_2(x,y)$ the places where the vertical line passing through $(x,y,0)$ enters and leaves the body D.

$$\iiint_D f(x,y,z)\,dV = \iint_R \left(\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz \right) dA$$

Then we can further reduce the double integral to an iterated integral as described in lecture 14.



$$\boxed{\iiint_D f(x,y,z)\,dV = \int_a^b \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz\,dy\,dx}$$

Examples: set up the limits of integration, don't compute the integral:

a) Find the centroid of the tetrahedron in the first octant bounded by the plane

$$\frac{x}{3} + \frac{y}{2} + \frac{z}{1} = 1$$

b) Find the volume of a sphere of radius a

c) Change the order of integration for

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{1} dz\,dy\,dx$$

to obtain an integral of the form $\iiint dx\,dy\,dz$.

The next goal is to see how polar coordinates can be set up in the 3-dim world. There are two analogues of them: cylindrical coordinates and spherical coordinates.

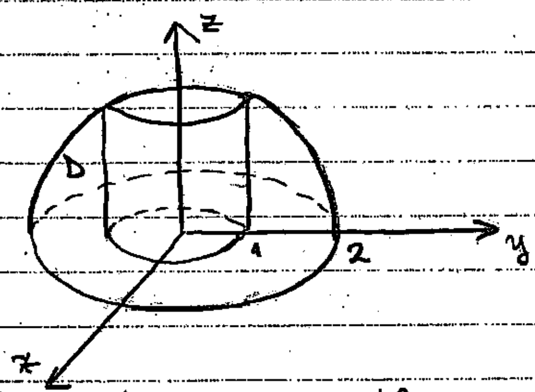
4. Cylindrical coordinates: are just an extension of polar coordinates from 2d to 3d by adding the z-coordinate.

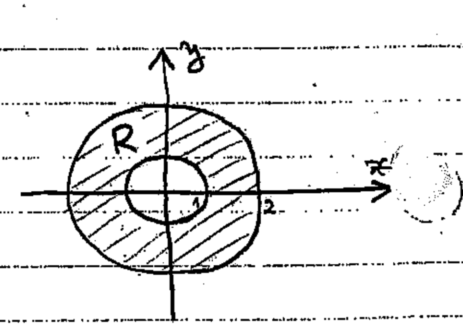
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

No surprise that $dV = dz \cdot dA = dz \cdot \underbrace{r\,dr\,d\theta}_{\text{from polar coordinates}}$

Examples

a) Set up the limits of integration and then find the centroid of the upper hemisphere of radius 2 from which a cylindrical hole of radius 1 was bored through the center.





Project D on the xy-plane. We get an annulus. This region in polar coordinates has limits

$$\int_0^{2\pi}\int_1^2 \cdots\, r\,dr\,d\theta$$