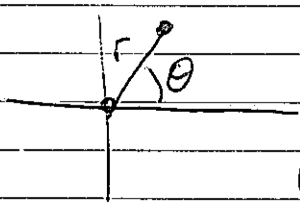
(The numerators of these fractions are the <u>moments</u> of mass of the thin plate w.r.t. the x- & y-axes, respectively.)

Note when $\delta =$ const, we've already discussed (back when we did Pappus' Theorem.)

Insert here note that
$$\int_c^d\int_a^b f(x)g(y)\,dx\,dy = \int_a^b f(x)\,dx \cdot \int_c^d g(y)\,dy$$

Next: <u>polar coordinates</u>.

Idea: locate point by distance from the origin (r) & angle w/ positive x-axis (θ)



First observation: not unique

$$(r,\theta) = (-r, \theta+\pi) = (r, \theta+2\pi) = (-r, \theta-\pi) = \dots$$

Also, $(0,\theta)$ is the origin for every angle θ.

Translating:
$$x = r\cos\theta \qquad r^2 = x^2+y^2$$
$$y = r\sin\theta \qquad \tan\theta = y/x$$

The graph of an equation in polar coordinates is just the set of points that satisfy the equation.

<u>Examples</u>: Graph:
- $r=1$
- $\theta = \pi/4$
- $r = 2\cos\theta$

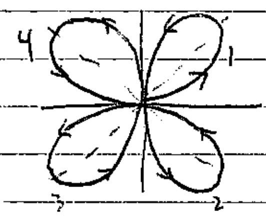
<u>Question</u>: Which of the following points lie on the curve with equation $r = \sin^2\theta$?
- $(2, \pi/4)$
- $(1, \pi/2)$
- $(0, 3\pi/2)$ (tricky!)

Question: identify the graph $r = 2\csc\theta$.

Question: give a polar equation for the rectangular equation $y = x^2$.

__Use__: some things are much easier to describe in polar coordinates than in rectangular coordinates: a simple example is the 4-leaf rose

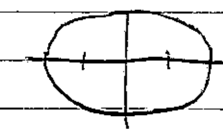
$r = a\sin 2\theta$



— imagine trying to give it in rectangular coordinates!

(Question: can we give it parametric coordinates?)

__Remark__: finding intersections of polar curves can be very tricky: consider the curves $r = 1 + \cos^2\theta$ & $r = -1 - \cos^2\theta$.

__Their graphs are identical__ ( __Note__: __Not__ an ellipse.)

Now, let's consider some of the classical problems of calculus for polar coordinates. __Arclength__:

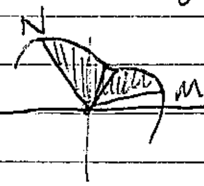
$$x = r\cos\theta \Rightarrow dx = dr\cdot\cos\theta - r\sin\theta\, d\theta$$
$$y = r\sin\theta \Rightarrow dy = dr\cdot\sin\theta + r\cos\theta\, d\theta$$

$$\begin{aligned}\Rightarrow ds^2 &= dx^2 + dy^2 \\ &= \left(dr^2\cos^2\theta - 2r\sin\theta\cos\theta\, dr\, d\theta + r^2\sin^2\theta\, d\theta^2\right) \\ &\quad + \left(dr^2\sin^2\theta + 2r\sin\theta\cos\theta\, dr\, d\theta + r^2\cos^2\theta\, d\theta^2\right) \\ &= dr^2 + r^2\, d\theta^2\end{aligned}$$

So, for example, arclength of the spiral $r = \theta^2$?

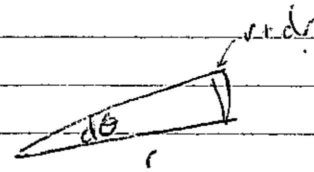
Areas in polar coordinates

Given a curve "around the origin," how much area in the sector it defines?



Introduce this the other way!

$$\text{Area} = \int_M^N dA$$

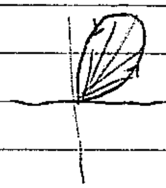


$$dA = \frac{1}{2} r^2 \, d\theta$$

$$\text{Area} = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 \, d\theta$$

Some care is necessary in choosing the right bounds.

Area inside one leaf of four-leaf rose $r = \sin 2\theta$?



$$\int_0^{\pi/2} \frac{1}{2} (\sin 2\theta)^2 \, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}\left(\frac{1-\cos 4\theta}{2}\right) d\theta$$

$$= \ldots$$

ended here

Sometimes polar coordinates make double integrals easier to compute (e.g., if region is most naturally described in polar).

In this case, our element of area is