Have seen that the equation of tangent plane to the surface given implicitly by $F(x,y,z)=0$ @ (x_0,y_0,z_0) has eqn $\vec{\nabla}F(x_0,y_0,z_0)\cdot\langle x-x_0, y-y_0, z-z_0\rangle = 0$

or, in the case $F(x,y,z) = z - f(x,y)$ (i.e., $z=f(x,y)$ is equation of surface)

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

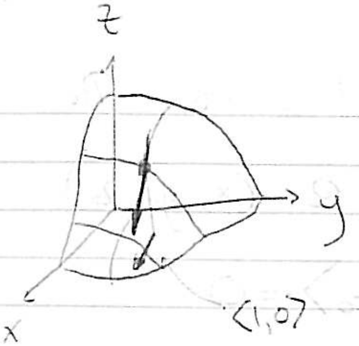
This is also the linear approximation to $f(x,y)$ near (x_0,y_0): $f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$, when $(x,y)\approx(x_0,y_0)$. (Need condition that f_x & f_y are continuous for these to be valid.)

Among other things, this gives us a good estimate of the sensitivity to the various variables, e.g., if

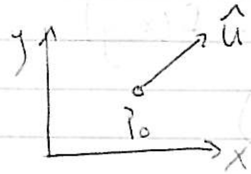
$f(x,y) = \dfrac{y}{x^2+1}$, is f more sensitive to x or to y @ $(x,y)=(1,2)$? What is the equation of the tangent plane to the surface $z=f(x,y)$ @ this point? $\vec{\nabla}f$?

<u>Directional derivatives</u>

We know partial derivatives give us rate of change w.r.t. one variable holding others constant. Thus, we can say for example that $\frac{\partial f}{\partial x}$ is the rate of change of f in the direction $\langle 1,0,\ldots\rangle$.



Directional derivative lets you measure change in any direction.

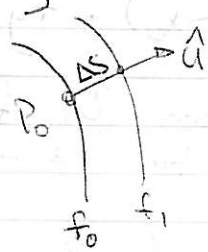


We want $\boxed{\lim_{\Delta s \to 0} \frac{\Delta f}{\Delta s}}$, where by Δf we mean $f|_{P_0 + \Delta s\,\hat{u}} - f|_{P_0}$.

small change in direction $\hat{u}$.

$$\boxed{\left.\frac{df}{ds}\right|_{\hat{u}}}$$

One way is to estimate from level curves



$$\left.\frac{df}{ds}\right|_{\hat{u}} \approx \frac{f_1 - f_0}{\Delta s} = \frac{\Delta f}{\Delta s}$$

Other way is to break $\Delta s\,\hat{u} = \langle \Delta x, \Delta y \rangle$ into components:

$$\Delta f \approx \left.\left(\frac{\partial f}{\partial x}\right)\right|_{P_0} \cdot \Delta x + \left.\left(\frac{\partial f}{\partial y}\right)\right|_{P_0} \cdot \Delta y$$

$$= \left\langle \left.\frac{\partial f}{\partial x}\right|_{P_0}, \left.\frac{\partial f}{\partial y}\right|_{P_0} \right\rangle \cdot \langle \Delta x, \Delta y \rangle.$$

Dividing by Δs gives

$$\frac{\Delta f}{\Delta s} \approx \vec{\nabla} f(P_0) \cdot \underbrace{\left\langle \frac{\Delta x}{\Delta s}, \frac{\Delta y}{\Delta s} \right\rangle}_{\hat{u}},$$

& taking the limit makes this

$$\left.\frac{df}{ds}\right|_{\hat{u}, P_0} = \vec{\nabla} f(P_0) \cdot \hat{u}$$

i.e. $\left.\frac{df}{ds}\right|_{\hat{u}} = \vec{\nabla} f \cdot \hat{u}$

Example: compute the directional derivative of $f(x,y) = \frac{y}{x^2+1}$ at the point $(1,2)$ in the directions $\langle 3,4\rangle$, $\langle 1,1\rangle$, $\langle -1,0\rangle$. Which one results in the largest change in f? What direction gives the steepest ascent from this point?

Maximizing & minimizing functions of several variables.

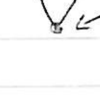
$P_0 = (x_0, y_0)$ local max or min of $w = f(x,y)$ $\Rightarrow$

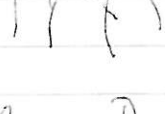
 horizontal tangent plane $\Leftrightarrow$ $w_x = 0$ & $w_y = 0$.

Analytically, (which covers case of more variables), if (x_0, y_0) is a local max (min) for $w = f(x,y)$ then x_0 is a local max (min) for $f(x, y_0)$ & y_0 is a local max (min) for $f(x_0, y)$.

Call points where $w_x = w_y = 0$ critical points (differs somewhat from our usage in 1-var case)

All local maxima & minima are among critical points

or vertices / other "bad pts" (analogue of cusps & discontinuities) $z = \sqrt{x^2+y^2}$  $(0,0)$ min, but z_x, z_y undefined.

Also have saddle points (critical pt but not max or min)  $z = x^2 - y^2$

& possibly maxima or minima along the boundary of a region (no longer just at endpoints)

Recall that to help us in 1-var case we had the 2^{nd} derivative test:

$$f' = 0 \quad \& \quad \begin{cases} f'' < 0 & \Rightarrow \text{max} \\ f'' > 0 & \Rightarrow \text{min} \\ f'' = 0 & ??? \end{cases}$$

What's the analogue of the 2^{nd} derivative here? The

<u>Hessian matrix</u> of 2^{nd} partials

$$H(f) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} \quad \left(\text{or} \quad \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix} \right)$$

If $\vec{\nabla} f = \vec{0}$ &
- $\det H > 0$, $f_{xx} < 0$, $f_{yy} < 0 \Rightarrow$ <u>max</u>
- $\det H > 0$, $f_{xx} > 0$, $f_{yy} > 0 \Rightarrow$ <u>min</u>
- $\det H < 0 \Rightarrow$ saddle
- $\det H = 0 \Rightarrow$????

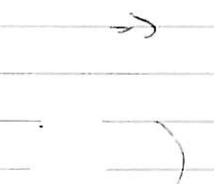
(note these are exhaustive)

<u>Examples</u>:
- $f(x,y) = xy - x^2 - y^2$
- $f(x,y) = 3xy - x^2 - y^2$
- $f(x,y) = 2xy - x^2 - y^2$

what are critical pts, etc.?

(Generalization of 2^{nd} derivative test is to "negative definite" & "positive definite" matrices: If $\det H \neq 0$, all eigenvalues of H are negative $\Rightarrow$ max; $\det H \neq 0$, all eigenvalues are positive $\Rightarrow$ min; $\det H \neq 0$, both negative & positive eigenvalues $\Rightarrow$ saddle point.)

One application: best-fit line:



$y = mx + b$ s.t. $D = \sum_{i=1}^{n} (mx_i + b - y_i)^2$ is minimized ?

("deviation")

What if we want to maximize a functions when some variables are inter-related? For example, we said that maxima & minima may occur on a boundary region — how do we cope with that?

Sometimes not so bad: e.g., find extrema of function $f(x,y) = e^{x+y} - 2e^x - e^{2y}$ subject to $0 \le x, y \le 2$.

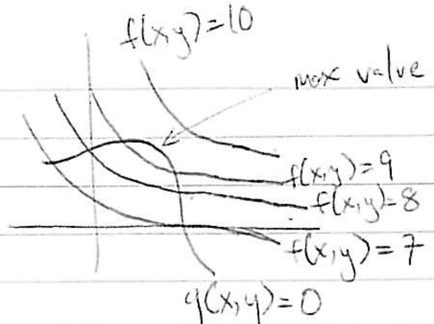
$f_x = e^{x+y} - 2e^x$ $\quad f_x = 0 \Rightarrow e^y = 2 \Rightarrow y = \ln 2$
$f_y = e^{x+y} - 2e^{2y}$ $\quad f_y = 0 \Rightarrow e^x = 2e^y = 4 \Rightarrow x = \ln 4$

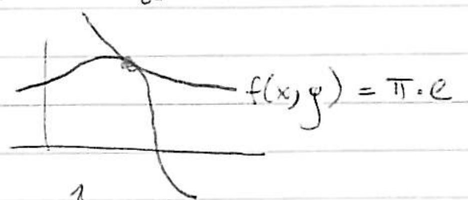
Now separately look along lines $x=0 \Rightarrow f(0,y) = e^y - e^{2y} - 2$
Then analyze this fn w/ 1-var techniques, then $x=2$, then $y=0$, then $y=2$ & eventually get the answer
(can let them do on own if desire; ans is that there is 1 crit pt but it's a saddle pt; max value is @ $(2, 2-\ln 2)$ & min value is at $(0,2)$.)

Now, this gets even worse if boundary of our region is some smooth curve (or some other equation constraining our variables, in case of more than 2). What to do?

Suppose we want to maximize a function $f(x,y)$ subject to the constraint $g(x,y) = 0$ (or any k) (e.g., to maximize some function over the unit circle: the $g(x,y) = x^2+y^2-1$.) Then consider overlaying this curve on the contour plot of $z = f(x,y)$:

max value between 9 & 10 ... where?





exactly when the two curves just touch

i.e., are tangent $\Rightarrow$ gradient vectors are parallel

$\Rightarrow \quad \vec{\nabla} f = \lambda \vec{\nabla} g$ at maximum (for some value λ.)

So, method of <u>Lagrange multipliers</u>: so find extremal points of $f(x,y,\ldots)$ subject to the constraint $g(x,y,\ldots) = k$,

constant

solve the system of equations

$$\boxed{\begin{aligned} f_x(x,y,\ldots) &= \lambda\, g_x(x,y,\ldots) \\ f_y(x,y,\ldots) &= \lambda\, g_y(x,y,\ldots) \\ &\vdots \\ \& \quad g(x,y,\ldots) &= k \end{aligned}}$$

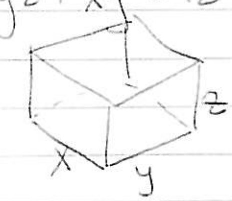
(for $\lambda, x, y, \ldots$)

& then test the various solutions to find the max'l & min'l values.

<u>E.g.</u> What is the largest open box that can be made from 12 m² of material?

Maximize $V = xyz$ subject to $g(x,y,z) = 2xz + 2yz + xy = 12$

$\vec{\nabla} V = \langle yz, xz, xy \rangle \qquad \vec{\nabla} g = \langle 2z+y, 2z+x, 2x+2y \rangle$



$$\vec{\nabla} V = \lambda \vec{\nabla} g \quad \Rightarrow \quad \begin{aligned} yz &= \lambda(2z+y) \\ xz &= \lambda(2z+x) \\ xy &= \lambda(2x+2y) \\ 2xz+2yz+xy &= 12 \end{aligned}$$

Note $\lambda \neq 0$ (else $xy = yz = xz = 0$, contradicting ~~third~~ fourth eqn). Then from first & second equations,

$$\lambda(2xz + xy) = \lambda(2yz + xy)$$
$$\Rightarrow \quad 2xz + xy = 2yz + xy$$
$$\Rightarrow \quad x = y$$

& from second & third,

$$\lambda(2yz + xy) = \lambda(2xz + 2yz)$$
$$\Rightarrow \quad 2yz + xy = 2xz + 2yz$$
$$\Rightarrow \quad y = 2z$$
$$\Rightarrow \quad x = y = 2z.$$

Then from fourth equation,

$$2xz + 2yz + xy = 12$$
$$\Rightarrow \quad 4z^2 + 4z^2 + 4z^2 = 12$$
$$\Rightarrow \quad z = 1, \quad x = y = 2$$

<u>Example 2:</u> Maximize the linear function $f(x,y,z) = 2x + 6y + 10z$ <u>on the sphere $x^2 + y^2 + z^2 = 35$.</u>

<u>Multiple constraints</u>

Suppose we want extreme values of $f(x,y,z)$ subject to constraints $g(x,y,z) = 0$ (or k_1) & $h(x,y,z) = 0$ (k_2).

Then solve instead the system

$$\vec{\nabla} f = \lambda \vec{\nabla} g + \mu \vec{\nabla} h$$
$$g = 0 \leftarrow (k_1)$$
$$h = 0 \leftarrow (k_2)$$

(& similarly for more variables or more constraints)

<u>E.g.</u> Maximize $x + 2y + 3z$ subject to $x - y + z = 1$ & $x^2 + y^2 = 1$.

$\Rightarrow$ Solve

$$1 = \lambda + 2x\mu$$
$$2 = -\lambda + 2y\mu$$
$$3 = \lambda$$
$$x - y + z = 1$$
$$x^2 + y^2 = 1$$

(etc.)